Convex valuations invariant under the Lorentz group

Semyon Alesker and Dmitry Faifman*

Abstract

We give an explicit classification of translation-invariant, Lorentz-invariant continuous valuations on convex sets. We also classify the Lorentz-invariant even generalized valuations.

1 Introduction

The main result of this paper is to give a complete classification of translation invariant continuous valuations on convex sets in $\mathbb{R}^n$ invariant under the connected component of the Lorentz group.

Let $\mathcal{K}^n$ denote the family of convex compact subsets of $\mathbb{R}^n$. A (convex) valuation is a functional $\phi\colon \mathcal{K}^n \to \mathbb{C}$ which satisfies the following additivity property

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}^n$. A valuation is called continuous if it is continuous with respect to the Hausdorff metric on $\mathcal{K}^n$.

Classification results are playing an important role in the valuations theory and its applications to integral geometry since the fundamental work of Hadwiger in the 1940's and 1950's. Probably the most famous result in the area is Hadwiger's characterization [14] of continuous valuations on convex subsets of a Euclidean space invariant under all isometries, i.e. translations and all orthogonal transformations, as linear combinations of intrinsic volumes (see [22] for this notion); the subgroup of orientation preserving isometries leads to the same list of invariant valuations. In recent years many new classification results have been obtained for various classes of valuations. Thus Klain [15] and Schneider [23] have classified continuous translation invariant valuations which are simple, i.e. vanish on convex sets of positive codimension. In [1] the first author have proven the following general results: let G be a compact subgroup of the linear group. The subspace of G-invariant translation invariant continuous valuations on convex sets is finite dimensional if and only if G acts transitively on the unit

*The authors were partially supported by ISF grants 701/08 and 1447/12

sphere; thus for such a group G one may hope to get certain finite classification. The problem to obtain such a classification is under investigation of a few people in recent year. Notice that the cases $G = O(n), SO(n)$ correspond to the Hadwiger theorem. The next interesting case $G = U(n)$ was classified explicitly in geometric terms by the first author [3] where also first applications to Hermitian integral geometry were obtained. More thorough and complete further study of $U(n)$-invariant valuations and Hermitian integral geometry was done by Bernig and Fu [9] and Fu [10]. Several other cases of compact groups acting transitively on the sphere were considered by Bernig [5], [6], [7].

At the same time other classes of valuations were studied under weaker assumptions on continuity but stronger assumptions on the symmetry group, which usually was either $GL_n(\mathbb{R})$ or $SL_n(\mathbb{R})$. Thus Ludwig and Reitzner [18] have characterized the affine surface area as the only (up to volume, Euler characteristic, and non-negative multiplicative factor) upper semi-continuous valuation invariant under affine volume preserving transformations. Other results on $SL_n(\mathbb{R})$-invariant valuations were obtained again by Ludwig and Reitzner [19]. Quite a few of classification results in a different but related direction of convex body valued valuations were obtained in [16], [17], [24], [25]; see also references therein.

Let us now discuss in greater detail the main results of the present paper. Let us fix on $\mathbb{R}^n$ the Minkowski metric, i.e. sign indefinite quadratic form Q of signature $(n-1,1)$. In coordinates it is given by $Q(x) = \sum_{i=1}^{n-1} x_i^2 - x_n^2$. Let $O(n-1,1)$ denote the group of all linear transformations of $\mathbb{R}^n$ preserving Q. It is well known that $O(n-1,1)$ has four connected components. Let us denote by $SO^+(n-1,1)$ the connected component of the identity. Throughout the article, we refer to $SO^+(n-1,1)$ as the Lorentz group.

Let us denote by $Val(\mathbb{R}^n)$ the space of all translation invariant continuous valuations on $\mathbb{R}^n$. For an integer k let us denote by $Val_k(\mathbb{R}^n)$ the subspace of k-homogeneous valuations (a valuation ϕ is called k-homogeneous if $\phi(\lambda K) = \lambda^k \phi(K)$ for any $\lambda \geq 0$ and any convex compact set K). McMullen's decomposition theorem [20] says that

$$Val(\mathbb{R}^n) = \oplus_{k=0}^n Val_k(\mathbb{R}^n). \tag{1}$$

$Val_k(\mathbb{R}^n)$ can be decomposed further with respect to parity:

$$Val_k(\mathbb{R}^n) = Val_k^{ev}(\mathbb{R}^n) \oplus Val_k^{odd}(\mathbb{R}^n),$$

where a valuation ϕ is called even (resp. odd) if $\phi(-K) = \phi(K)$ (resp. $\phi(-K) = -\phi(K)$) for any K.

It is easy to see that $Val_0(\mathbb{R}^n)$ is spanned by the Euler characteristic, i.e. valuation which is equal to 1 on any convex compact set. By a theorem of Hadwiger [14], $Val_n(\mathbb{R}^n)$ is spanned by the Lebesgue measure.

We denote by $Val(\mathbb{R}^n)^{SO^+(n-1,1)}$ the subspace of $SO^+(n-1,1)$-invariant valuations, and similarly for subspaces of given parity and homogeneity. McMullen's decomposition (1) immediately implies

$$Val(\mathbb{R}^n)^{SO^+(n-1,1)} = \oplus_{k=0}^n (Val_k^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)} \oplus Val_k^{odd}(\mathbb{R}^n)^{SO^+(n-1,1)})$$

Our first main result classifies odd $SO^+(n-1,1)$-invariant valuations.

Theorem 1.1. *For $0 \leq k \leq n$, $k \neq n-1$, $Val_k^{odd}(\mathbb{R}^n)^{SO^+(n-1,1)} = 0$.*
For $k = n-1$,

$$\dim Val_k^{odd}(\mathbb{R}^n)^{SO^+(n-1,1)} = \begin{cases} 1, & n \geq 3 \\ 2, & n = 2 \end{cases}$$

The last space will be described explicitly.

The proof of this result relies on Schneider's imbedding theorem and makes use of Lie group continuous cohomology as one of the tools to show that the Schneider bundle has no non-zero continuous $SO^+(n-1,1)$-invariant sections for $1 \leq k \leq n-2$.

Our second main result classifies even $SO^+(n-1,1)$-invariant valuations. Notice first of all that by the above discussion 0- and n-homogeneous valuations are proportional to the Euler characteristic and Lebesgue measure, respectively. In particular they are even and $SO^+(n-1,1)$-invariant.
For the remaining degrees of homogeneity, namely $1 \leq k \leq n-1$, the classification consists of two parts. First, we define and classify the invariant generalized valuations. Roughly speaking, this is the completion of the space of smooth valuations with respect to a certain weak topology that is defined using the product structure on valuations (see subsection 4.2 for precise definitions). The space of generalized valuations naturally contains the continuous valuations as a dense subspace. We then analyze which of the invariant generalized valuations are in fact continuous. The following two theorems summarize our results:

Theorem 1.2. *For all $1 \leq k \leq n-1$, the space of k-homogeneous, even, $SO^+(n-1,1)$-invariant generalized valuations is 2-dimensional. Those spaces will be described explicitly.*

Theorem 1.3. *For $1 \leq k \leq n-2$, $Val_k^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)} = 0$. For $k = n-1$, $\dim Val_k^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)} = 2$. This space will be described explicitly.*

Let us remark that the generalized Lorentz-invariant odd valuations remain to be classified.

The plan of the classification is as follows: First, we study $SO^+(n-1,1)$-invariant continuous sections of the Klain bundle. For any $1 \leq k \leq n-1$ we get a 2-dimensional space of $SO^+(n-1,1)$-invariant continuous sections. By McMullen's theorem, this finishes the classification of continuous $(n-1)$-homogeneous even valuations. For the remaining $1 \leq k \leq n-2$, it turns out that those sections correspond to generalized valuations, which are not continuous. We construct the corresponding generalized valuations explicitly (section 4), and then proceed to show that they are discontinuous by proving that they cannot be evaluated on the double cone (sections 3,5). This last part of analysis involves lengthy technical arguments. Another difficulty in comparison to the case of groups transitive on spheres is that $SO^+(n-1,1)$-invariant valuations do not have to be smooth in the sense of [2].

Finally, we give some applications of the classification. One is the explicit construction of a continuous section of Klain's bundle that lies in the closure of Klain's imbedding of the continuous valuations, but outside the image of the imbedding. The non-closedness of the image was proved very recently by Parapatits and Wannerer in [21] using different methods. Another corollary is the non-extendibility by continuity of the Fourier transform from smooth to continuous valuations, which also was proved in [21] using different methods.

Acknowledgement. We are grateful to José Miguel Figueroa-O'Farrill who has explained to us Proposition 2.7 on computation of continuous group cohomology.

2 Finding the Lorentz-invariant continuous sections of Klain's and Schneider's bundles

Let us introduce the notation used throughout the paper. For a linear space W, $Vol(W)$ will denote the 1-dimensional space of volume forms on W, and $D(W)$ the 1-dimensional space of densities on W. $Gr(W,k)$ is the Grassmannian of k-subspaces of W. The signature of a quadratic form Q will be denoted sign$Q$; We write $SO^+(n-1,1)$ for the identity component of the full Lorentz group $O(n-1,1)$. If a norm is given on W, $S(W)$ denotes the unit sphere in W. For a vector bundle E over a manifold M, $\Gamma^{\pm\infty}(M,E)$, or sometimes simply $\Gamma^{\pm\infty}(E)$, will denote the space of smooth resp. generalized sections.

It is well-known that the even continuous valuations naturally form a $GL(n)$-equivariant subspace of the continuous sections of Klain's bundle, which is the line bundle of densities on k-dimensional linear subspaces of $\mathbb{R}^n$, over $Gr(n,k)$. A similar result holds for odd continuous valuations; the precise description is given below. The definitions here are more technical and will be recalled later. To find all Lorentz-invariant valuations, we begin by determining all the invariant sections of those two bundles.

In the following, V stands for $\mathbb{R}^n$. Fix two symmetric 2-forms: Euclidean $\langle u,v\rangle = \sum_{j=1}^n u_j v_j$, and Lorentzian $Q(u,v) = \sum_{j=1}^{n-1} u_j v_j - u_n v_n$. Let (e_j) be the standard basis, and $\zeta(v) := \langle v, e_n\rangle$. The unit $n\times n$ matrix is denoted I.

2.1 Klain's bundle $K^{n,k}$

Let γ_n^k be the tautological vector bundle over $Gr(n,k)$, so that the fiber over $\Lambda \in Gr(\mathbb{R}^n,k)$ is simply Λ; and $K^{n,k}$ is the bundle of densities on its fibers, which is naturally a $GL(n)$-line bundle.The Euclidean structure defines a density in every subspace, i.e. we have a global section $Area \in \Gamma(K^{n,k})$, $Area$ is the only $SO(n)$-invariant continuous section (up to scale), and it defines a trivialization of the bundle. Denote by $SO^+(n-1,1)\subset GL(n)$ the connected component of the identity in the group of isometries of Q. We will study $SO^+(n-1,1)$-invariant continuous sections of $K^{n,k}$.

Proposition 2.1. *Given a Lorentz-orthogonal family $(v_1,...,v_k)$ s.t. $Q(v_i)=1$ for $i\le k-1$, and $Q(v_k)=\pm1$, and denoting $z_j=\zeta(v_j)$, one has*

$$Area(v_1,...,v_k)^2 = \begin{cases} 1+2\sum_{j=1}^k z_j^2, \ Q(v_k)=1 \\ 2(z_k^2-\sum_{j=1}^{k-1} z_j^2)-1, \ Q(v_k)=-1 \end{cases}$$

Proof. Use the identity

$$Area(v_1,...,v_k)^2 = det(\langle v_i,v_j\rangle) = \det(Q(v_i,v_j)+2z_iz_j) = \det(I_\pm + 2zz^T)$$

where $I_\pm$ is a $k\times k$ diagonal matrix with entries $Q(v_1),...,Q(v_k)$, and $z=(z_1,...,z_k)^T$. The remaining verification is straightforward. Q.E.D.

Proposition 2.2. *Given $T\in SO^+(n-1,1)$, and $\Lambda\in Gr(n,k)$ generic (i.e., Q restricted to Λ is non-degenerate), if $T(\Lambda)=\Lambda$ then $|\det T|_\Lambda|=1$.*

Proof. Since $Q|_\Lambda$ is non-degenerate, and $T\in GL(\Lambda)$ preserves Q, it follows that $|\det T|_\Lambda|=1$. Q.E.D.

Proposition 2.3. *The space of $G=SO^+(n-1,1)$-invariant continuous sections of $K^{n,k}$ is 2-dimensional*

Proof. 1. The orbits of the action of G on $Gr(n,k)$ are characterized by the signature of the restriction of Q. The open orbits are $M_+ = \{\Lambda : \mathrm{sign}Q|_\Lambda = (k,0)\}$ and $M_- = \{\Lambda : \mathrm{sign}Q|_\Lambda = (k-1,1)\}$. Together, $M_+ \cup M_-$ are dense in $Gr(n,k)$. The remaining orbit is $M_0 = \{\Lambda : signQ|_\Lambda = (k-1,0)\}$ (there are no 2 non-proportinal Q-orthogonal vectors on the light cone).

2. Choose some fixed $\Lambda_+ \in M_+$ and $\Lambda_- \in M_-$, and fix arbitrary densities on them. By Proposition 2.2, they extend to an invariant section μ of $M_+ \cup M_-$. It remains to verify that μ admits a continuous $G-$invariant extension to all $Gr(n,k)$. Let us show that $\mu(\Lambda) \to 0$ as $\Lambda \to M_0$. For this, it is enough to take a Q-orthonormal basis of Λ, denoted $v_1, ..., v_k$ and show that $Area(v_1, ..., v_k)^2 \to \infty$.

3. First, assume $M_+ \ni \Lambda \to M_0$.

Write $z = (z_1, ..., z_k)$. Define ϵ by $\langle P_\Lambda e_n, e_n\rangle = \cos(\pi/4+\epsilon)|P_\Lambda e_n|$, where P_Λ is the (Euclidean) projection onto Λ. We assume $Q(v_j) = 1$ for $1 \le j \le k$, Write $P_\Lambda e_n = \sum \alpha_j v_j$. Then $\langle P_\Lambda e_n - e_n, v_i\rangle = 0$, for all i, i.e. $(I + 2zz^T)(\alpha) = z$. By Sherman-Morrison [26] formula,

$$(I + 2zz^T)^{-1} = I - \frac{2zz^T}{1 + 2z^Tz} I$$

We will denote $A = Area(v_1, ..., v_k)^2$, $B = z^Tz = z_1^2 + ... + z_{k-1}^2 + z_k^2$. By Proposition 2.1, $A = 1 + 2B$. Then

$$\alpha = z - \frac{2zz^Tz}{1 + 2z^Tz} = \frac{1}{A} z$$

Let us write $\cos^2(\pi/4+\epsilon) = 1/2 - \delta$. Then

$$\langle P_\Lambda e_n, e_n\rangle^2 = \cos^2(\pi/4+\epsilon)|P_\Lambda e_n|^2 \Rightarrow \zeta(P_\Lambda e_n)^2 = (1/2-\delta)\Big(Q(P_\Lambda e_n) + 2\zeta(P_\Lambda e_n)^2\Big) \Rightarrow$$

$$\Rightarrow (\sum \alpha_j z_j)^2 = (1/2-\delta)(\sum_{j=1}^{k} \alpha_j^2) + (1-2\delta)(\sum \alpha_j z_j)^2$$

$$\Rightarrow 2\delta(\sum \alpha_j z_j)^2 = (1/2-\delta)(\sum_{j=1}^{k} \alpha_j^2)$$

Note that $\sum \alpha_j z_j = A^{-1}(z_1^2 + ... + z_{k-1}^2 + z_k^2) = \frac{B}{A} = \frac{A-1}{2A} = \frac{1}{2} - \frac{1}{2A}$, and $\sum_{j=1}^{k} \alpha_j^2 = \frac{B}{A^2} = \frac{A-1}{2A^2}$. Thus

$$\delta(1 - \frac{1}{A})^2 = (1/2-\delta)\frac{1}{A}(1 - \frac{1}{A}) \Rightarrow \frac{1}{A} = \frac{\delta}{1/2-\delta}(1 - 1/A)$$

$$\Rightarrow A = \frac{1}{2\delta} = \frac{1}{\sin 2\epsilon}$$

Thus $Area(v_1, ..., v_k) = \frac{1}{|\sin 2\epsilon|^{1/2}} \to \infty$ as $\delta \to 0$, i.e. $\mu(\Lambda) \to 0$ as $M_+ \ni \Lambda \to M_0$. This proves the existence of a section supported on M_+.

4. Now assume $M_- \ni \Lambda \to M_0$. Write $z = (z_1, ..., z_k)$. Let $\langle P_\Lambda e_n, e_n\rangle = \cos(\pi/4-\epsilon)|P_\Lambda e_n|$, where P_Λ is the orthogonal projection onto Λ. We assume $Q(v_j) = 1$ for $1 \le j \le k-1$, $Q(v_k) = -1$. Write $P_\Lambda e_n = \sum \alpha_j v_j$. Then $\langle P_\Lambda e_n - e_n, v_i\rangle = 0$, for all i, i.e. $(I_- + 2zz^T)(\alpha) = z$. By Sherman-Morrison,

$$(I_- + 2zz^T)^{-1} = I_- - \frac{2I_-zz^TI_-}{1 + 2z^TI_-z}$$

We will denote $\tilde{z} = I_- z$. Again using Proposition 2.1, we write $B = z^T I_- z = z_1^2 + ... + z_{k-1}^2 - z_k^2$, $A = Area(v_1, ..., v_k)^2 = -1 - 2B$. Then

$$\alpha = (I_- + 2zz^T)^{-1} z = I_- z - \frac{2}{1+2B} I_- z z^T I_- z = \tilde{z} - \frac{2B}{1+2B} I_- z = \frac{1}{1+2B} \tilde{z}$$

That is,

$$\alpha = -\frac{1}{A} \tilde{z}$$

Let us write $\cos^2(\pi/4 - \epsilon) = 1/2 + \delta$. Then

$$\langle P_\Lambda e_n, e_n \rangle^2 = \cos^2(\pi/4 - \epsilon) |P_\Lambda e_n|^2 \Rightarrow \zeta(P_\Lambda e_n)^2 = (1/2 + \delta) \Big(Q(P_\Lambda e_n) + 2\zeta(P_\Lambda e_n)^2 \Big) \Rightarrow$$

$$\Rightarrow (\sum \alpha_j z_j)^2 = (1/2 + \delta)(\sum_{j=1}^{k-1} \alpha_j^2 - \alpha_k^2) + (1 + 2\delta)(\sum \alpha_j z_j)^2$$

$$\Rightarrow -2\delta (\sum \alpha_j z_j)^2 = (1/2 + \delta)(\sum_{j=1}^{k-1} \alpha_j^2 - \alpha_k^2)$$

Note that $\sum \alpha_j z_j = -A^{-1}(z_1^2 + ... + z_{k-1}^2 - z_k^2) = -\frac{B}{A} = \frac{A+1}{2A} = \frac{1}{2} + \frac{1}{2A}$, and $\sum_{j=1}^{k-1} \alpha_j^2 - \alpha_k^2 = \frac{B}{A^2} = -\frac{A+1}{2A^2}$. Thus

$$\delta (1 + \frac{1}{A})^2 = (1/2 + \delta) \frac{1}{A} (1 + \frac{1}{A}) \Rightarrow \frac{1}{A} = \frac{\delta}{1/2 + \delta} (1 + 1/A)$$

$$\Rightarrow A = \frac{1}{2\delta} = \frac{1}{\sin 2\epsilon}$$

Again $Area(v_1, ..., v_k) = \frac{1}{|\sin 2\epsilon|^{1/2}} \to \infty$ as $\delta \to 0$, i.e. $\mu(\Lambda) \to 0$ as $M_- \ni \Lambda \to M_0$. Thus there is an invariant section supported on M_-, Q.E.D.

Corollary 2.4. *The space $Val_{n-1}^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)}$ is 2-dimensional, and consists of non-smooth sections. It is spanned by f_S and f_T (standing for space-like and time-like) given by*

$$f_T(K) = \int_{S^{n-1} \cap \{Q \geq 0\}} \sqrt{|\sin 2\epsilon|} d\sigma_K(\omega)$$

and similarly

$$f_S(K) = \int_{S^{n-1} \cap \{Q \leq 0\}} \sqrt{|\sin 2\epsilon|} d\sigma_K(\omega)$$

where ϵ denotes the angle between ω and the light cone, and $\sigma_K(\omega)$ is the surface area measure of K.

2.1.1 Geometrical interpretations of the space $Val_{n-1}^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)}$

This purpose of this subsection is to provide some geometrical intuition into the valuations that we constructed. It will not be used in the rest of the paper.

Let us denote $H^\pm = \{x \in \mathbb{R}^n : Q(x,x) = \pm 1\}$. Both H^+ and H^- inherit a Lorentzian resp. Riemannian metric from $(\mathbb{R}^n, Q)$. Then $H^- \subset (\mathbb{R}^n, Q)$ is the Minkowski model of hyperbolic space, and similarly H^+ is the $(n-2,1)$ de Sitter space. The valuations in $Val_{n-1}^{ev}(\mathbb{R}^n)^{SO^+(n-1,1)}$ can be interpreted as the surface area of K with respect to $H^\pm$ in the following sense:

Define the support functions $h_{H^+}, h_{H^-} : S^{n-1} \to \mathbb{R}$ by setting $h_{H^\pm}(\theta)$ equal to the distance from the origin of the hyperplane P_θ with Euclidean normal

equal to θ that is tangent to H^+ (resp. H^-). If no such hyperplane exists, the value of $h_{H^\pm}(\theta)$ is set to 0. Denoting by $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ the elevation angle on S^{n-1} relative to the spacelike coordinate hyperplane $(x_1, ..., x_{n-1})$, these functions are given explicitly by

$$h_{H^+}(\omega) = \begin{cases} \sqrt{|\cos 2\alpha|} & |\alpha| \leq \pi/4 \\ 0 & |\alpha| < \pi/4 \end{cases}$$

$$h_{H^-}(\omega) = \begin{cases} \sqrt{|\cos 2\alpha|} & |\alpha| \geq \pi/4 \\ 0 & |\alpha| < \pi/4 \end{cases}$$

Then we may think of f_T informally as a mixed volume:

$$f_T(K) = V(K[n-1], H^+[1]) = \int_{S^{n-1}} h_{H^+}(\omega) d\sigma_K(\omega)$$

and similarly

$$f_S(K) = V(K[n-1], H^-[1]) = \int_{S^{n-1}} h_{H^-}(\omega) d\sigma_K(\omega)$$

Another very similar description is the following. Assume K is smooth. The boundary ∂K inherits from $(\mathbb{R}^n, Q)$ a smooth field of quadratic forms on all tangent spaces. Then $f_S(K)$ is the total volume of the space-like part of ∂K (that is, where the form is positively defined), and similarly $f_T(K)$ is the volume of the time-like part.

There is also a relation between the $(n-1)$-homogeneous Lorentz-invariant valuations, and the surface area in hyperbolic and de Sitter spaces. More precisely, f_T and f_S correspond to the surface area on H^- and H^+, respectively, in the following sense. For a set $A \subset H^\pm$, define $C_A = \{tx : 0 \leq t \leq 1, x \in A\}$ the cone with base A. Denote by $Area_{H^\mp}$ the hyperbolic\de Sitter area on $H^\mp$, and $\phi_H^\mp$ is either f_T or f_S, respectively. Observe that while C_A is not a convex body, one can nevertheless compute f_S or f_T on C_A at least when A is piecewise geodesic (and so given by a finite collection of intersections of $H^\pm$ with hyperplanes in $\mathbb{R}^n$), simply by applying the explicit formulas of Corollary 2.4.

Proposition 2.5. *Let $A \subset H^\pm$ be a polytope. If $A \subset H^+$, we further assume it has spacelike boundary. Then*

$$Area_{H^\pm}(\partial A) = \phi_H^\pm(C_A)$$

Proof. An $(n-2)$-dimensional face F of A lies on $\Lambda \cap H^\pm$ for $\Lambda \in Gr(n, n-1)$. By additivity of both sides, it suffices to verify that $Area_{H^\pm}(F) = \phi_H^\pm(C_F)$. For H^+, by our assumption Λ is space-like, so the statement is simply that the cone measure on the sphere $\Lambda \cap H^+$ coincides with the spherical volume on it. For H^-, Λ is necessarily timelike, and it is again well-known (or easily checked) that the cone measure of the hyperboloid $\Lambda \cap H^-$ coincides with the hyperbolic volume. Q.E.D.

2.2 Schneider's bundle $S^{n,k}$

For every non-oriented subspace $\Omega \subset V$ of dimension $k+1$, consider the bundle of densities on the tautological bundle over the space of k-dimensional coori-ented subspaces $\Lambda \subset \Omega$, denoted $\widetilde{K}^{k+1,k}(\Omega)$. Let $\Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))$ denote the

space of global sections which are odd w.r.t. coorientation reversal of Λ, i.e. $\mu(\overline{\Lambda}) = -\mu(\Lambda)$.

There is a $k+1$-dimensional linear subspace $L(\Omega) \subset \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))$, consisting of sections that are defined by elements of Ω^*. The space $L(\Omega) \simeq \Omega \otimes D(\Omega)$ is defined as follows: For any k-dimensional $\Lambda \subset \Omega$, $\Lambda^\perp \subset \Omega^*$, and

$$D(\Lambda) \otimes D(\Omega/\Lambda) = D(\Omega)$$

so

$$D(\Lambda) = D(\Omega) \otimes D(\Omega/\Lambda)^* = D(\Omega) \otimes D(\Lambda^\perp)$$

Any $v \in \Omega$ defines a density $|v|$ on $\Lambda^\perp \subset \Omega^*$ for all Λ, and so we define $\mu_{v\otimes d}(\Lambda) = \mathrm{sign}(v, \Lambda)|v| \otimes d \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))$ for $v \otimes d \in \Omega \otimes D(\Omega)$. Here $\mathrm{sign}(v, \Lambda) = \pm 1$ is determined by the coorientation of Λ (and $\mathrm{sign}(v, \Lambda) = 0$ for $v \in \Lambda$). The image of the map $v \otimes d \mapsto \mu_{v\otimes d}$ is denoted. Let $F_\Omega = \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))/L(\Omega)$ be the quotient.

Schneider's bundle $S^{n,k}$ consists of the base space $Gr(V, k+1)$, and fiber F_Ω. The topology can be introduced by fixing an orthonormal basis on V, which gives the identifications $\Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega)) = C_{odd}(S(\Omega))$, $L(\Omega) = \Omega^* = \Omega$, and $F_\Omega = \Omega^\perp \subset C_{odd}(S(\Omega))$, the orthogonal complement taken in the $L^2_{odd}(S(\Omega))$ norm . In particular, F_Ω inherits an inner product induced from $L^2_{odd}(S(\Omega))$. Note that any global section $s \in \Gamma(S^{n,k})$ gives a continuous in Ω family of functions $\mu_\Omega \in C_{odd}(S(\Omega))$. Schneider's imbedding gives for every odd k-homogeneous valuation a global section $s \in \Gamma(S^{n,k})$.

However, for a G-equivariant section s (where G is some group), this lift is not a-priori G-equivariant. This is because the lift is defined by an arbitrarily chosen Euclidean structure.

We will classify the $G = SO^+(n-1, 1)$-invariant sections of $S^{n,k}$.

Theorem 2.6. *There are no odd $SO^+(n-1,1)$-invariant k-homogeneous valuations for $1 \le k \le n-2$. For $k = n-1$ and $n \ge 3$, the space $Val_k^-(\mathbb{R}^n)^{SO^+(n-1,1)}$ is 1-dimensional. Finally, the space $Val_1^-(\mathbb{R}^2)^{SO^+(1,1)}$ is 2-dimensional.*

Proof. Let s be such a section. We assume at first that $k \le n-2$.

0. Denote $M_+ = \{\Omega \in Gr(V, k+1) : Q|_\Omega > 0\}$, $M_- = \{\Omega \in Gr(V, k+1) : \mathrm{sign}Q|_\Omega = (k, 1)\}$, $M_0 = \{\Omega \in Gr(V, k+1) : \mathrm{sign}Q|_\Omega = (k, 0)\}$. Those are the orbits of G as it acts on $Gr(V, k+1)$. We will write $\mathrm{Stab}(\Omega) \subset G$ for the stabilizer of Ω, and $\mathrm{Stab}^+(\Omega) = \{T \in \mathrm{Stab}(\Omega) : \det T|_\Omega = 1\}$ is the orientation-preserving subgroup of $\mathrm{Stab}(\Omega)$.

1. Observe that s necessarily vanishes on M_+: Fix some $\Omega \in M_+$. Take the Euclidean structure on Ω to be $Q|_\Omega$, and then obtain a lift $\mu_\Omega \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))$ of s_Ω which is $\mathrm{Stab}(\Omega)$-invariant. Since $\mathrm{Stab}(\Omega)$ is transitive on $Gr^+(\Omega, k)$ (in fact, it is transitive even under $\mathrm{Stab}^+(\Omega)$), $\mu_\Omega(\Lambda) = \mu_\Omega(\overline{\Lambda})$ for all Λ, so $\mu_\Omega = 0$ on Ω. Thus $s = 0$ on M_+, and by continuity, it follows that s vanishes on M_0.

2. Now consider M_-. For any fixed $\Omega \in M_-$ one has a $\mathrm{Stab}(\Omega)$-invariant element $s_\Omega \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))/L(\Omega)$. Since $H^1_c(\mathrm{Stab}^+(\Omega); \Omega) = 0$ (see 2.2.1 below for the computation), we can choose $\mu_\Omega \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))^{\mathrm{Stab}^+(\Omega)}$ lifting s_Ω, and then, possibly after averaging with $g_0\mu_\Omega$ (which also lifts s_Ω) where $g_0 \in \mathrm{Stab}(\Omega)$

is orientation-reversing, we may assume $\mu_\Omega \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))^{\mathrm{Stab}(\Omega)}$. In fact, if we fix any $\Omega_0 \in M_-$ and the corresponding $\mu_0 = \mu_{\Omega_0}$, then for any $g \in G$ one can take $\mu_{g\Omega_0} = g_*\mu_0 \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(g\Omega_0))^{\mathrm{Stab}(g\Omega_0)}$. We thus get a G-invariant lift of s to a continuous family of sections of $\Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega))$ over $\Omega \in M_-$.

3. We want to inspect μ_0 more closely. The group $\mathrm{Stab}(\Omega_0)$ has the following open orbits as it acts on the cooriented hyperplanes $\Lambda \subset \Omega_0$: Ignoring the coorientation, there are two non-oriented open orbits, consisting of X_+, the Q-positive Λ and X_-, those Λ with signature $(k-1,1)$.
An orientation of $\Lambda \in X_+$ is fixed under $g \in \mathrm{Stab}(\Omega_0) \cap \mathrm{Stab}(\Lambda)$ iff the orientation of Ω_0 is fixed, so coorientation is always preserved. Thus X_+ splits into two orbits X_1 and X_2 when coorientation is accounted for.
On the other hand, X_- constitutes a single orbit including coorientation. There are two cases to consider: when $k=1$, $\Lambda \in X_-$ is a time-like line and so has its orientation preserved under the action of $g \in \mathrm{Stab}(\Omega_0) \cap \mathrm{Stab}(\Lambda)$, while the orientation of Ω_0 can be preserved or reversed (since $\dim \Omega_0 = k+1 \le n-1$).
If $k \ge 2$, the verification is also straightforward: one can again reverse the orientation of Ω_0 while kee ping the orientation of Λ.
We conclude that $\mu_0(\Lambda) = 0$ for all $\Lambda \in X_-$: Indeed, since μ_0 is odd, $\mu_0(\overline{\Lambda}) = -\mu_0(\Lambda)$; but both $\Lambda, \overline{\Lambda}$ lie in the same $\mathrm{Stab}(\Omega_0)$-orbit, so $\mu_0(\Lambda) = 0$.

4. Observe that on any $\Lambda \subset \Omega_0$ which is Q-degenerate, $\mu_0(\Lambda) = 0$ by continuity from X_+.
So μ_0 is uniquely defined (since it is odd, and through G-invariance) by a density $\mu_+ \in D(\Lambda_+)$ for some Q - positive subspace $\Lambda_+ \subset \Omega_0$.
Note that, as was the case with Klain's bundle, any such μ_+ extends to a continuous $\mu_0 \in \Gamma_{odd}(\widetilde{K}^{k+1,k}(\Omega_0))^{\mathrm{Stab}(\Omega_0)}$, and then to a family μ_Ω for $\Omega \in M_-$

5. Let us show that μ_Ω has a limit μ_∞ in $\Gamma_{odd}\left(\widetilde{K}^{k+1,k}(\Omega_\infty)\right)$ as $\Omega \to \Omega_\infty \in M_0$.
Assume for simplicity that some orientation is fixed on Ω_∞. For every Q-positive oriented k-subspace $\Lambda \subset V$, choose $\Omega_\Lambda = \Lambda \oplus \langle e_n \rangle$ with the natural orientation, and $\mu(\Lambda) = \mu_{\Omega_\Lambda}(\Lambda) \in D(\Lambda)$. The family μ_Ω is thus equivalent to a G-equivariant collection $\mu(\Lambda)$ of densities on all Q-positive k-dimensional oriented subspaces Λ, s.t. $\mu(\overline{\Lambda}) = -\mu(\Lambda)$. Then for $M_- \ni (\Omega_t, \Lambda_t) \to (\Omega_\infty, \Lambda_\infty)$, either $\mu(\Lambda_t) \to \mu(\Lambda_\infty)$ when Λ_∞ is Q-positive by continuity of μ_Ω, or $\mu(\Lambda_t) \to 0 \in D(\Lambda_\infty)$. Thus μ_∞ is well-defined. The limit of $[\mu_\Omega]$ in $\Gamma_{odd}\left(\widetilde{K}^{k+1,k}(\Omega_\infty)\right)/L(\Omega_\infty)$ is therefore $[\mu_\infty]$, and it must vanish as $\Omega \to \Omega_\infty \in M_0$, by continuity of s and since s vanishes on M_+. Therefore, μ_∞ is a linear section that vanishes on all Q-degenerate k-subspaces. This is equivalent to a linear functional on $\mathbb{R}^{k+1}$ that vanishes on the light cone. So $\mu_\infty = 0$, implying $\mu_\Omega = 0$.

We conclude that when $k \le n-2$, there are no G-invariant sections of Schneider's bundle. It follows there are no non-trivial continuous, odd, $k-$homogeneous valuations.

Now assume $k = n-1$. Again since $H^1_c(G;V) = 0$, we may lift s to an invariant section $\mu \in \Gamma_{odd}(\widetilde{K}^{n,n-1}(V))^G$.
If $n \ge 3$, as in step 3 above, μ must vanish on mixed-signature subspaces; and μ is determined by its value μ_+ on one positive subspace. Unlike the case $k \le n-2$, there are no other restrictions: any μ_+ extends to a global section μ, as was the case with Klain's bundle.
If $n = 2$, as in step 2 above μ is determined by two independent densities $\mu_+(\Lambda_+)$ and $\mu_-(\Lambda_-)$; and any two such densities give a continuous μ_Ω as with Klain's bundle.

For $k = n-1$, Schneider's imbedding is really just the McMullen characterization of odd $n-1$-homogeneous valuations, i.e. the imbedding is an isomorphism, concluding the classification of $n-1$ -homogeneous invariant valuations. Q.E.D.

2.2.1 Computation of the continuous Lie group cohomology

The main result of this section was explained to us by José Miguel Figueroa-O'Farrill. For the relevant definitions, see [11]. We need to compute the continuous cohomolgy of $G = SO^+(n-1,1)$ with coefficients in the standard representation $V = \mathbb{R}^n$. Specifically, we will show

Proposition 2.7. *The first continuous group cohomology $H^1_c(G;V)$ vanishes.*

Proof. Consider $SO(n-1) \subset G$ - the maximal compact subgroup. By the Hochschild-Mostow Theorem ,

$$H^1_c(G;V) = H^1(\mathfrak{so}(n-1,1), \mathfrak{so}(n-1); V)$$

We will write $\mathfrak{g} = \mathfrak{so}(n-1,1)$ and $\mathfrak{h} = \mathfrak{so}(n-1)$. Under the action of $\mathfrak{h}$, $V = W \oplus T$ where $W = \mathbb{R}^{n-1}$ is the standard representation of $SO(n-1)$ (corresponding to the space coordinate hyperplane), and $T = \mathbb{R}$ is the trivial representation (corresponding to the time axis of V). Also, the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ admits the decomposition $\mathfrak{g} = \mathfrak{h} \oplus W$ where the inclusion $i : W \hookrightarrow \mathfrak{g}$ is given by

$$v \mapsto \begin{pmatrix} 0_{(n-1)\times(n-1)} & v_{(n-1)\times 1} \\ v^T_{1\times(n-1)} & 0 \end{pmatrix}$$

Note also that $[\mathfrak{h}, W] = W$. Now

$$C^0(\mathfrak{g}, \mathfrak{h}; V) = \{v \in V : \mathfrak{h}v = 0\} = T = \mathbb{R}$$

while

$$C^1(\mathfrak{g}, \mathfrak{h}; V) = \{f \in \mathrm{Hom}(\mathfrak{g}, V) : f(\mathfrak{h}) = 0, f([h,g]) = hf(g)\, \forall g \in \mathfrak{g}, h \in \mathfrak{h}\} =$$

$$= \{f \in \mathrm{Hom}(W, V) :, f([h,w]) = hf(w)\, \forall w \in W, h \in \mathfrak{h}\} =$$

$$= \{f \in \mathrm{Hom}(W, W) :, f([h,w]) = hf(w)\, \forall w \in W, h \in \mathfrak{h}\}$$

that is, $C^1(\mathfrak{g}, \mathfrak{h}; V) = Hom(W,W)^{\mathfrak{h}}$. This space consists of scalar operators when $\dim W \geq 3 \iff n \geq 4$, and of complex-linear operators when $n = 3$ and $W = \mathbb{R}^2 = \mathbb{C}$. The differential map $d_1 : C^0(\mathfrak{g}, \mathfrak{h}; V) \to C^1(\mathfrak{g}, \mathfrak{h}; V)$ is nonzero: taking some $t \in T$, $d_1 t(w) = -i(w)(t) = -tw$ so $d_1 t \neq 0$. Thus $\dim \mathrm{Im}(d_1) = 1$. For $n \geq 4$, $\dim C^1(\mathfrak{g}, \mathfrak{h}; V) = 1$ and it follows that $H^1(\mathfrak{g}, \mathfrak{h}; V) = 0$.

When $n = 3$, $\dim C^1(\mathfrak{g}, \mathfrak{h}; V) = 2$ while $d_1(C^0(\mathfrak{g}, \mathfrak{h}; V)) \subset \mathrm{Ker}(d_2) \subset C^1(\mathfrak{g}, \mathfrak{h}; V)$. We should check whether $d_2 = 0$. It is enough to check the value of d_2 on some non-scalar operator, say $J \in \mathrm{Hom}(W,W)^{\mathfrak{h}}$ which corresponds to $\frac{\pi}{2}$-rotation. Let w_1, w_2 be the standard basis of W. Then

$$d_2 J(g_1, g_2) = J([g_1, g_2]) - g_1 J(g_2) + g_2 J(g_1)$$

Since $\mathfrak{h} \subset \mathrm{Hom}(d_2 J)$ and $\mathfrak{g} = \mathfrak{h} \oplus W$, $d_2 J \neq 0 \iff d_2 J(w_1, w_2) \neq 0$. Now

$$[i(w_1), i(w_2)] = J \in \mathfrak{h}$$

so $J([i(w_1), i(w_2)]) = 0$. And

$$-i(w_1)J(i(w_2)) + i(w_2)J(i(w_1)) = i(w_1)w_1 + i(w_2)w_2 = (0,0,2)^T$$

so $d_2 J \neq 0$.

Thus $\dim \mathrm{Ker} d_2 = 1$ also for $n = 3$, and $H^1(\mathfrak{so}(n-1,1), \mathfrak{so}(n-1); V) = 0$ for all n. Q.E.D.

Now consider the exact sequence $0 \to L(V) \to \Gamma_{odd}(\widetilde{K}^{n,n-1}(V)) \to F_V \to 0$ where $L(V)$ is the space of linear sections on V (an n-dimensional space), and it is G-isomorphic to V. We have the long exact sequence of cohomology

$$0 \to L(V)^G \to \Gamma_{odd}(\widetilde{K}^{n,n-1}(V))^G \to F_V^G \to H^1(G; L(V)) = 0$$

it follows that every G-invariant section of F_V lifts to a G-invariant section of $\Gamma_{odd}(\widetilde{K}^{n,n-1}(V))$.

3 Computing valuations on $SO(n-1)$-invariant unconditional bodies

Definition 3.1. The k-support function of a body $K \subset \mathbb{R}^n$, denoted $h_k(\Lambda; K) \in C(Gr(n, n-k))$, is the k-volume of the projection of K to $\Lambda^\perp$.

Let $L \subset \mathbb{R}^2$ be a convex, unconditional body. Denote $L^n \subset \mathbb{R}^n$ its rotation body around the vertical axis, namely

$$L^n = \{(\omega x, y) | \omega \in S^{n-2}, (x,y) \in L\}$$

Denote also $h_k(\alpha; L) = h_k(\alpha; L^{k+1})$ for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$: it is obvious that the k-support function of L^n is $SO(n-1)$-invariant for all $1 \le k \le n-1$, and so it really is a function of α. Here $\alpha = 0$ corresponds to a vertical hyperplane. By abuse of notation, we consider $h_k(\alpha; L)$ to be a function both on the unit circle S^1 and on the sphere $S^k \subset \mathbb{R}^{k+1}$; we will write $h_k(\alpha)$ or $h_k(\omega)$ when we need to emphasize that the domain is S^1, resp. S^k. Denoting $R_{k+1} \in O(k+1)$ the reversal of time direction and $G_{k+1} = \langle SO(k), R_{k+1} \rangle \subset O(k+1)$, it is obvious that L^n is G_n-invariant.

Proposition 3.2. *L^n is a convex unconditional body, and $h_k(\alpha; L^n) = h_k(\alpha, L)$ for all $n > k$. Any G_n-invariant convex body equals L^n for some L as above.*

Proof. The Minkowski functional of L^n is $p_n(\omega x, y) = \|(x,y)\|_L$ for $x, y \in \mathbb{R}$, $\omega \in S^{n-1}$. Let us verify it is convex:

$$p_n(\omega_1 x_1, y_1) + p_n(\omega_2 x_2, y_2) = \|(x_1, y_1)\|_L + \|(x_2, y_2)\|_L \ge \|(|x_1| + |x_2|, |y_1| + |y_2|)\|_L$$

while

$$p_n((\omega_1 x_1, y_1) + (\omega_2 x_2, y_2)) = p_n(\omega_1 x_1 + \omega_2 x_2, y_1 + y_2) =$$
$$= \|(|\omega_1 x_1 + \omega_2 x_2|, y_1 + y_2)\|_L \le \|(|x_1| + |x_2|, |y_1| + |y_2|)\|_L$$

by unconditionality of L. The unconditionality of L^n is obvious. Now $h_k(\alpha; L^n)$ can be computed as follows. Let $e_1, ..., e_n$ be the standard basis, and define $\Omega = \mathrm{Span}\{e_1, ..., e_k, e_n\}$. Let $\Lambda_\alpha \subset \Omega$ be a k-dimensional subspace forming angle α with the spacelike coordinate hyperplane. Then $h_k(\alpha; L^{k+1}) = h_k(\alpha; \Omega \cap L^n) = \mathrm{vol}_k(\mathrm{Pr}_{\Lambda_\alpha}(\Omega \cap L^n))$ and by unconditionality of L, $\mathrm{Pr}_\Omega(L^n) = L^n \cap \Omega$ so

$$h_k(\alpha, L^n) = \mathrm{vol}_k(\mathrm{Pr}_{\Lambda_\alpha}(L^n)) = \mathrm{vol}_k(\mathrm{Pr}_{\Lambda_\alpha}\mathrm{Pr}_\Omega(L^n)) = h_k(\alpha; L^{k+1})$$

Finally, given a G_n- invariant convex body K, it is immediate that its 2-dimensional x_1-x_n section L will be convex and unconditional, and $K = L^n$, concluding the proof. Q.E.D.

Remark 3.3. It follows that $L \mapsto L^n$ is a Hausdorff homeomorphism between the spaces of 2-dimensional unconditional convex bodies and $SO(n-1)$-invariant, unconditional convex bodies.

Recall the cosine transform $T_k : C^\infty(S^k) \to C^\infty(S^k)$ given by

$$T_k(f)(y) = \int_{S^k} f(x)|\langle x, y\rangle|dx$$

is a self-adjoint isomorphism when restricted to even functions, and extends to an isomorphism of generalized even functions. It is well-known that $T_k(\sigma_k(\omega; L)) = h_k(\omega; L)$ where $\sigma_k \in C(S^k)^*$ is the surface-area measure of L^{k+1}.

Lemma 3.4. *If $f \in C^\infty(\mathbb{R})$ is even, then $f(|x|) \in C^\infty(\mathbb{R}^n)$.*

Proof. This is because $f(x) = g(x^2)$ for $g \in C^\infty[0, \infty)$. Q.E.D.

For the following, we recall the definition of Sobolev spaces. On the linear space $\mathbb{R}^k$, denote $f \mapsto \hat{f}$ the Fourier transform, and the p-Sobolev space is the completion of $C_c^\infty(\mathbb{R}^k)$ w.r.t. the norm $\|f\|_{L_p^2} = \|\hat{f}(\omega)(1 + |\omega|^p)\|_{L^2}$. For a compact smooth manifold X, $L_p^2(X) \subset C^{-\infty}(X)$ is defined by some choice of a finite atlas $\{U_\alpha\}$ for X and an attached partition of unity $\{\rho_\alpha\}$:

$$L_p^2(X) = \{\sum_\alpha \rho_\alpha f_\alpha : f_\alpha \in L_p^2(U_\alpha)\}$$

The resulting space $L_p^2(X)$ is independent of the choices made.

Proposition 3.5. *For all $k \geq 1$ and $\epsilon > 0$, $h_k(\omega; L) \in L^2_{\frac{3}{2}-\epsilon}(S^k)$. If $h_1(\alpha; L)$ is smooth in a neighborhood of the poles and the equator, then $h_k(\omega; L) \in L^2_{\frac{k}{2}+1-\epsilon}(S^k)$ is smooth near the poles, and $h_k(\alpha; L) \in L^2_{\frac{k}{2}+1-\epsilon}(S^1)$.*

Proof. Denote

$$\Box = \frac{1}{2\omega_{k-1}}(\Delta + k) : C^\infty_{even}(S^k) \to C^\infty_{even}(S^k)$$

where ω_{k-1} is the surface area of S^{k-1}. It is an invertible differential operator of order 2. Let $\mathcal{R}_k : C^\infty_{even}(S^k) \to C^\infty_{even}(S^k)$ denote the spherical Radon transform, which is an invertible Fourier integral operator of order $-\frac{k-1}{2}$ (see [13]). Then (see [12])

$$\Box T_k = \mathcal{R}_k \iff T_k = \Box^{-1}\mathcal{R}_k \tag{2}$$

Therefore, the cosine transform T_k is an invertible (on even functions) Fourier integral operator of order $-\frac{k+3}{2}$, and it respects Sobolev spaces, i.e. for all $s \in \mathbb{R}$

$$T_k : L_s^2(S^k) \to L^2_{s+\frac{k+3}{2}}(S^k)$$

is an isomorphim. In particular, T_1 is invertible by a differential operator followed by a $\frac{\pi}{2}$-rotation.

For the first part, note that the surface area measure $\sigma_k \in C(S^k)^* \subset L^2_{-\frac{k}{2}-\epsilon}(S^k)$, so $h_k(\omega; L) = T_k(\sigma_k) \in L^2_{\frac{3}{2}-\epsilon}(S^k)$.

For the second part, note that $\sigma_1 = T_1^{-1}(h_1) \in C(S^1)^* \subset L^2_{-\frac{1}{2}-\epsilon}(S^1)$ is smooth in a neighborhood of the equator and of the poles of S^1, since h_1 is smooth there, and by eq. 2. Let $\sigma_k = \pi^*\sigma_1$ be the surface area measure of L^{k+1}, where $\pi : S^k \to S^k/SO(k-1)$. Then σ_k is smooth near the poles from unconditionality of L and Lemma 3.4, so $\sigma_k \in L^2_{-\frac{1}{2}-\epsilon}(S^k)$; also σ_k is smooth near the equator $S^{k-1} \subset S^k$. Therefore, $h_k(\omega; L) = T_k(\sigma_k) \in L^2_d(S^k)$ where $d = -\frac{1}{2} - \epsilon + \frac{k+3}{2} = \frac{k}{2} + 1 - \epsilon$, and also h_k is smooth near the poles. Then $h_k(\alpha; L)$, which can be obtained by taking a vertical 2-dimensional restriction of $h_k(\omega; L)$, lies in $L^2_d(S^1)$ and is smooth near the poles, as required. Q.E.D.

Remark 3.6. It follows that under the assumptions of Proposition 3.5, $h_k \in C^{\lfloor \frac{k}{2} \rfloor}(S^1)$, and if $K_n \to K$ in the Hausdorff topology s.t. $h_1(\bullet; K_n)$ and $h_1(\bullet; K)$ are as above, then also $h_k(\alpha; K_n) \to h_k(\alpha; K)$ in the $C^{\lfloor \frac{k}{2} \rfloor}(S^1)$ topology.

Proposition 3.7. *Let $\phi \in Val_k^+(\mathbb{R}^n)^{SO(n-1)}$ be a continuous k-homogeneous even valuation such that $\phi(K^n) = \int_{S^1} f h_k(\alpha; K)$ for $SO(n-1)$-invariant convex bodies K^n with smooth $h_k(\bullet; K)$, where $f \in C^{-\infty}_{even}(S^1)$. Then $\phi(K^n) = \int_{S^1} f h_k(\alpha; K)$ for all $SO(n-1)$-invariant symmetric convex bodies K^n such that sing-supp$(h_k(\alpha; K))$ and sing-supp(f) are disjoint one from another, and sing-supp$(h_k(\alpha; K))$ is disjoint from the poles.*

Proof. Denote $G = SO(k+1)$, $H = SO(k)$. Write $S^k = H\backslash G$ for the space of orbits under left action. Let $d\mu$ be the Haar probability measure on G, $d\sigma$ the pushforward to S^k. Fix a positive approximate identity $F_N \in C^\infty(S^k)^H$ supported near the north pole (identified with its H- bi-invariant pullback to G). It can be obtained by fixing an approximate identity $\tilde{F}_N$ on G, and then taking

$$F_N(g) = \int_{H \times H} \tilde{F}_N(h_1 g h_2) dh_1 dh_2$$

Note that $F_N(g) = F_N(g^{-1})$ by bi-invariance of F_N, and since $\langle gH, H\rangle = \langle H, g^{-1}H\rangle$ (considered as points on the sphere).
Convolution of functions is defined by

$$u * v(x) = \int_G u(g) v(g^{-1}x) d\mu(g) = \int_G v(g) u(xg^{-1}) d\mu(g)$$

so that $(L_h u) * v = L_h(u * v)$ and $R_h(u * v) = u * R_h v$ (here $L_h$and R_h denote the left and right actions respectively). In particular, for $u \in C^\infty(S^k)$, $v \in C^\infty(G)$, $u * v \in C^\infty(S^k)$, and if v is right H-invariant, so is $u * v$. The following properties hold:
1. Convolution with F_N on either side is self adjoint: for $u, v \in C^\infty(S^k)$, $\langle F_N * u, v\rangle = \langle u, F_N * v\rangle$and $\langle u * F_N, v\rangle = \langle u, v * F_N\rangle$. For instance,

$$\langle F_N * u, v\rangle = \int_G d\mu(x) v(Hx) \int_G d\mu(g) u(Hg) F_N(xg^{-1}) = \int_{G\times G} d\mu(x) d\mu(g) v(Hx) u(Hg) F_N(xg^{-1})$$

and we can exchange x and g since $F_N(xg^{-1}) = F_N(gx^{-1})$. Similarly,

$$\langle u * F_N, v\rangle = \int_G d\mu(x) v(Hx) \int_G d\mu(g) u(Hg) F_N(g^{-1}x) = \int_{G\times G} d\mu(x) d\mu(g) v(Hx) u(Hg) F_N(g^{-1}x)$$

2. For $u \in C^\infty(S^k)$, one has $F_N * u \to u$ in $C^\infty(S^k)$. For $u \in C^\infty(G/H)$, $u * F_N \to u$.

$$F_N * u(x) = \int_{H\times H} dh_1 dh_2 \int_G \tilde{F}_N(h_1 g h_2) u(g^{-1}x) dg = \int_{H\times H} dh_1 dh_2 \int_G \tilde{F}_N(h_1 g) u(h_2^{-1} g^{-1} x) dg$$

by left H-invariance of u, this equals

$$\int_H dh \int_G \tilde{F}_N(hg)u(g^{-1}x)dg = \int_H dh\tilde{F}_N * u(hx)dh = \int_H L_h(\tilde{F}_N * u)(x)dh$$

since $L_h(\tilde{F}_N * u)(x) \to L_h u(x) = u(x)$ in $C^\infty(G)$, we conclude that $\int_H L_h(\tilde{F}_N * u)(x)dh \to u$ in $C^\infty(S^k)$. Similarly, for $u \in C^\infty(G/H)$,

$$u*F_N(x) = \int_{H\times H} dh_1 dh_2 \int_G \tilde{F}_N(h_1 g h_2)u(xg^{-1})dg = \int_H dh_2 \int_G \tilde{F}_N(gh_2)u(xg^{-1})dg =$$

$$= \int_H dh \int_G \tilde{F}_N(g)u(xh^{-1}g^{-1})dg = \int_H R_h(u * \tilde{F}_N)(x)dh$$

and again

$$R_h(u * \tilde{F}_N) \to R_h u = u$$

implying the statement.
3. For $u \in C^{-\infty}(S^k)$, $F_N * u \to u$ for $u \in C^{-\infty}(S^k)$ and $u * F_N \to u$ for $u \in C^{-\infty}(G/H)$. This is a direct consequence of properties 1 and 2.
4. For $u \in C^{-\infty}(S^k)$, $T_k(u * F_N) = T_k(u) * F_N$. It is enough by self-adjointness of T_k and the convolution operator to verify this for $u \in C^\infty(S^k)$:

$$T_k(u * F_N)(x) = \int_{S^k} dy|\langle x, y\rangle| \int_G dg F_N(g)u(yg^{-1}) =$$

$$= \int_G dg F_N(g) \int_{S^k} u(yg^{-1})|\langle x, y\rangle|dy = \int_G dg F_N(g) \int_{S^k} u(y)|\langle xg^{-1}, y\rangle|dy =$$

$$= \int_G dg F_N(g) T_k u(xg^{-1}) = T_k u * F_N(x)$$

Note that $F_N * u \in C^{-\infty}(S^k)^H$ whenever $u \in C^{-\infty}(S^k)^H$.

Let $\sigma_k \in C^{-\infty}(S^k)^H$ be the surface area measure of K^{k+1}. Then by Minkowski's theorem, $\sigma_k * F_N$ is the surface area measure of a sequence of H-invariant bodies denoted K_N^{k+1} s.t. $K_N \to K$, therefore also $K_N^n \to K^n$ and $\phi(K_N^n) \to \phi(K^n)$. On the other hand,

$$T(\sigma_k * F_N) = h_k(\bullet; K) * F_N$$

so

$$\phi(K_N^n) = \int_{S^1} f \cdot T(\sigma_k * F_N)d\alpha = \int_{S^1} f(\alpha) \cdot (h_k(\bullet; K) * F_N)(\alpha)d\alpha$$

Choose a cut-off function $\chi \in C^\infty(S^1)^{\mathbb{Z}_2}$ (the action is reflection w.r.t. the vertical axis, note that χ induces a smooth $H-$invariant function on S^k, also denoted χ) such that $\chi(\alpha)h_k(\bullet; K) \in C^\infty(S^k)$ and $(1 - \chi(\alpha))f(\alpha) \in C^\infty(S^1)$, and $\chi = 1$ in a neighborhood of the poles. Now we can restrict $\chi(\alpha)h_k(\bullet; K)$ to a smooth function on S^1, and

$$\chi(\alpha)(F_N * h_k(\bullet; K))(\alpha) \to \chi(\alpha)h_k(\alpha; K)$$

in $C^\infty(S^k)$ and also in $C^\infty(S^1)$ (by restriction). Then

$$\int_{S^1} f(\alpha)(h_k(\bullet; K) * F_N)(\alpha)d\alpha = \int_{S^1} f(\alpha)\Big(\chi(\alpha)(h_k(\bullet; K) * F_N)(\alpha)\Big)d\alpha +$$

$$+ \int_{S^1} \Big((1 - \chi(\alpha))f(\alpha)\Big)(h_k(\bullet; K) * F_N)(\alpha)d\alpha$$

The first summand converges to

$$\int_{S^1} f(\alpha)\chi(\alpha)h_k(\alpha;K)d\alpha$$

Also, $(1-\chi(\alpha))f(\alpha)$ can be pulled back to a smooth function on S^k since $1-\chi=0$ near the poles. In particular, we will have

$$\int_{S^1}\Big((1-\chi(\alpha))f(\alpha)\Big)(h_k(\bullet;K)*F_N)(\alpha)d\alpha \to \int_{S^1}\Big((1-\chi(\alpha))f(\alpha)\Big)h_k(\alpha;K)(\alpha)d\alpha$$

And so the sum converges to $\int_{S^1} f(\alpha)h_k(\alpha;K)d\alpha$, as required. Q.E.D.

4 Finding the generalized invariant valuations

From now on, $n\geq 3$, and $G=SO^+(n-1,1)$. Let us recall some definitions and facts and introduce notation.

Consider the bundle $E^{n,k}$ over $Gr(V,n-k)$ with fiber over $\Lambda\in Gr(V,n-k)$ equal to $E^{n,k}|_\Lambda = D(V/\Lambda)\otimes D(T_\Lambda Gr(V,n-k))$. We will sometimes refer to it as the Crofton bundle, and we call its (generalized) sections (generalized) Crofton measures. Also, recall Klain's bundle $K^{n,k}$ over $Gr(V,k)$, that has fiber $D(\Lambda)$ over $\Lambda\in Gr(n,k)$. Klain's imbedding $Kl: Val_k^{ev}(V)\to\Gamma(K^{n,k})$ is $GL(V)$-equivariant, and maps smooth valuations to smooth sections, see [1].

Observe there is a natural bilinear non-degenerate pairing

$$\Gamma^{\pm\infty}(E^{n,k})\times\Gamma^{\mp\infty}(K^{n,n-k})\to D(V)$$

The $GL(V)-$equivariant cosine transform $T_{n-k,k}:\Gamma^\infty(E^{n,k})\to\Gamma^\infty(K^{n,k})$ is given by

$$T_{n-k,k}(\gamma)(D_\Lambda)=\int_{\Omega\in Gr(n,n-k)}\gamma\otimes Pr_{V/\Omega}(D_\Lambda)$$

where $D_\Lambda\subset\Lambda$ is some symmetric convex body. We will write $T_{n-k,k}:C^\infty(Gr(n,n-k))\to C^\infty(Gr(n,k))$ also for the cosine transform after a Euclidean trivialization, and also $T_{n-k,k}:\Gamma^{-\infty}(E^{n,k})\to\Gamma^{-\infty}(K^{n,k})$ for the adjoint operator to $T_{k,n-k}:\Gamma^\infty(E^{n,n-k})\to\Gamma^\infty(K^{n,n-k})$. It extends the cosine transform on smooth sections.

4.1 Some representation theory

We make use of the following facts (see [4]):

1. The highest weights of $SO(n)$ are parametrized by sequences of integers $\lambda=(\lambda_1,...,\lambda_{\lfloor\frac{n}{2}\rfloor})$ with $\lambda_1\geq...\geq\lambda_{\lfloor\frac{n}{2}\rfloor}\geq 0$ for odd n, and $\lambda_1\geq...\geq\lambda_{\lfloor\frac{n}{2}\rfloor-1}\geq|\lambda_{\lfloor\frac{n}{2}\rfloor}|$ for even $n>2$.
2. The irreducible components of $C^\infty(Gr(n,k))$ (considered as a representation of $SO(n)$) are of multiplicity one, with highest weights $\lambda\in\Lambda_k^+\cap\Lambda_{n-k}^+$. Here $\Lambda_j=\{\lambda:\lambda_i=0\,\forall i>j,\,\lambda_i\equiv 0 \mod 2\,\forall i\}$.
3. The image of $T_k: C^\infty(Gr(n,n-k))\to C^\infty(Gr(n,k))$ consists of representations with highest weights $\lambda\in\Lambda_k^+\cap\Lambda_{n-k}^+$, $|\lambda_2|\leq 2$. The kernel is thus $\mathrm{Ker}T_k=\oplus\rho_\lambda$ with $\lambda\in\Lambda_k^+\cap\Lambda_{n-k}^+$, $|\lambda_2|\geq 4$. The image of T_k is closed.

4. The irreducible representations of $SO(n)$ which contain an $SO(n-1)$-invariant element are precisely those corresponding to spherical harmonics. Their highest weight is $(d, 0, ..., 0)$ (for degree d spherical harmonics). The spherical harmonics appearing in $C^\infty(G(n, n-k))$ are precisely those of even degree d.

5. In particular, $C^\infty(Gr(n, n-k))^{SO(n-1)} \cap \mathrm{Ker} T_{n-k,k} = 0$. Thus

$$T_{n-k,k} : C^\infty(Gr(V, n-k))^{SO(n-1)} \to C^\infty(Gr(V, k))^{SO(n-1)} \quad (3)$$

is an isomorphism: It is injective and has dense image (by Schur's Lemma), and also

$$T_{n-k,k}\Big((C^\infty(Gr(V, n-k)))^{SO(n-1)}\Big) = \Big(T_{n-k,k}(C^\infty(Gr(V, n-k)))\Big)^{SO(n-1)} \quad (4)$$

implying the image is closed. Equation 4 holds because $T_{n-k,k}$ obviously maps $SO(n-1)$-invariant vectors to $SO(n-1)$-invariant vectors, and if $v \in T_{n-k,k}(C^\infty(Gr(V, n-k)))$ is $SO(n-1)$-invariant, then $v = T_{n-k,k}u$ for some $u \in C^\infty(Gr(V, n-k))$ such that $v = T_{n-k,k}(gu)$ for all $g \in SO(n-1)$, implying $v = T_{n-k,k}(\int_{SO(n-1)} gu \cdot dg)$.

6. In particular,

$$T_{n-k,k} : C^{-\infty}(Gr(V, n-k))^{SO(n-1)} \to C^{-\infty}(Gr(V, k))^{SO(n-1)}$$

is also an isomorphism, since $T_{n-k,k}$ is a symmetric opeator (after the obvious identification $Gr(V, k) = Gr(V, n-k)$).

Note that the action of $SO(n-1)$ on $\Gamma^{-\infty}(E^{n,k})$ and $\Gamma^{-\infty}(K^{n,k})$ (after a Euclidean trivialization) and on $C^{-\infty}(Gr(V, n-k))$ resp. $C^{-\infty}(Gr(V, k))$ coincides. We deduce the following

Corollary 4.1. *The map*

$$T_{n-k,k} : \Gamma^{-\infty}(E^{n,k})^{SO^+(n-1,1)} \to \Gamma^{-\infty}(K^{n,k})^{SO^+(n-1,1)}$$

is injective.

Let us prove the following

Proposition 4.2. $C^\infty(Gr(n, n-k)) \cap T_{n-k,k}(C^{-\infty}(Gr(n, n-k))) = C^\infty(Gr(n, n-k))$

Proof. Assume $h(\Lambda) = T_{k,n-k}(\sigma)$ for some $\sigma \in C^{-\infty}(Gr(n, k))$ and $h \in C^\infty(Gr(n, n-k))$. Choose an approximate identity $\mu_N \in \mathcal{M}^\infty(SO(n))$. Then $T_{k,n-k}(\sigma * \mu_N) = T_{k,n-k}(\sigma) * \mu_N = h * \mu_N \to h$ in the C^∞-topology. Since $\sigma * \mu_N \in C^\infty(Gr(n, k))$, and the image of $T_{k,n-k}$ is closed in the C^∞ topology, it follows that $h \in T_{n-k,k}(C^\infty(Gr(n, k)))$, as claimed. Q.E.D.

4.2 Lorentz-invariant generalized valuations

The space $Val_k^{ev,-\infty}(V)$ of generalized k-homogeneous even valuations is defined by

$$Val_k^{ev,-\infty}(V) = \Big(Val_{n-k}^{ev,\infty}(V)\Big)^* \otimes D(V) = \Big(Val_{n-k}^{ev,\infty}(V) \otimes D(V)^*\Big)^*$$

By the Alesker-Poincare duality, there is a natural inclusion $Val_k^{ev,\infty}(V) \subset Val_k^{ev,-\infty}(V)$.

Let us write this inclusion explicitly. Recall that a Crofton measure $\mu_\phi \in \Gamma^\infty(Gr(V,n-k),E^{n,k})$ for $\phi \in Val_k^{ev,\infty}(V)$ is any section such that $T_{n-k,k}(\mu_\phi) = Kl(\phi)$, which always exists by [4]. It is equivalent to a smooth, translation-invariant measure on the affine Grassmannian $\overline{Gr}(V,n-k)$.

For $\phi \in Val_k^{ev,\infty}(\mathbb{R}^n)$ and $\psi \in Val_{n-k}^{ev,\infty}(\mathbb{R}^n)$, the duality map is given by

$$\langle \phi, \psi \rangle = \langle Kl(\phi), \mu_\psi \rangle$$

Equivalently,

$$\langle \phi, \psi \rangle(\bullet) = \int_{\overline{Gr}(V,k)} \phi(\bullet \cap E) d\mu_\psi(E) \in D(V)$$

We have the surjective map

$$Cr_k : \Gamma^\infty(E^{n,n-k}) \to Val_{n-k}^{ev,\infty}(V)$$

given by

$$Cr_k(s)(K) = \int_{\Lambda \in Gr(V,k)} s(Pr_{V/\Lambda}(K))$$

We will need the following

Claim 4.3. Let $T : X \to Y$ be a bounded linear map between Frechet spaces X, Y such that $Im(T) \subset Y$ is closed. Then $Im(T^*) \subset X^*$ is also closed.

Proof. By Banach's open mapping theorem, $T : X/\mathrm{Ker}(T) \to \mathrm{Im}(T)$ is an isomorphism of Frechet spaces. Therefore, $T^* : \mathrm{Im}(T)^* \to (X/\mathrm{Ker}(T))^* = \mathrm{Ker}(T)^\perp$ is also an isomorphism. It remains to observe that $T^* : Y^* \to X^*$ factorizes as $Y^* \twoheadrightarrow \mathrm{Im}(T)^* \simeq \mathrm{Ker}(T)^\perp \hookrightarrow X^*$ and the last inclusion is closed. Q.E.D.

Proposition 4.4. *There is a unique extension by continuity of Klain's imbedding,* $Kl_k : Val_k^{ev,-\infty}(V) \to \Gamma^{-\infty}(K^{n,k})$*, which is an imbedding with closed image.*

Consider the adjoint map of Cr_k:

$$Cr_k^* : Val_k^{ev,-\infty}(V) \otimes D(V)^* \to \Gamma^{-\infty}(K^{n,k}) \otimes D(V)^*$$

which gives a map

$$A : Val_k^{ev,-\infty}(V) \to \Gamma^{-\infty}(K^{n,k})$$

s.t. $Cr_k^* = A \otimes Id$. Let us verify that A extends Klain's imbedding $Kl_k : Val_k^{ev,\infty}(V) \to \Gamma^\infty(K^{n,k})$. For $\gamma \in \Gamma^\infty(E^{n,n-k})$, one has the obvious Crofton measure $\mu_{Cr_k(\gamma)} = \gamma$, so for all $\psi \in Val_k^{ev,\infty}(V)$

$$A(\psi)(\gamma) = \langle Cr_k(\gamma), \psi \rangle = \int_{Gr(V,k)} \mu_{Cr_k(\gamma)} Kl_k(\psi) =$$

$$= \int_{Gr(V,k)} \gamma Kl_k(\psi) = \langle \gamma, Kl_k(\psi) \rangle$$

as required. Moreover, $\mathrm{Ker}A = 0$, since Cr_k is surjective, and by Claim 4.3 the image of A is closed.

Proposition 4.5. *The map Cr_k admits a unique extension by continuity* $Cr_k : \Gamma^{-\infty}(E^{n,n-k}) \to Val_{n-k}^{ev,-\infty}(V)$ *which is surjective. It holds that* $Kl_{n-k} \circ Cr_k = T_{k,n-k}$.

Consider the dual to Klain's imbedding $Kl_k : Val_k^{ev,\infty}(V) \to \Gamma^\infty(K^{n,k})$, tensored with the identity on $D(V)$: It is given by

$$B : \Gamma^{-\infty}(E^{n,n-k}) \to Val_{n-k}^{ev,-\infty}(V)$$

where

$$B(s)(\psi) = \langle s, Kl_k(\psi)\rangle$$

for all $\psi \in Val_k^{ev,\infty}(V)$. Then B extends the Crofton surjection: for $\gamma \in \Gamma^\infty(K^{n,n-k})$ and $\psi \in Val_k^{ev,\infty}(V)$,

$$B(\gamma)(\psi) = \langle \gamma, Kl_k(\psi)\rangle = \langle Cr_k(\gamma), \psi\rangle$$

Let us verify it is surjective: the image of B is dense since Kl_k is injective. The image of B is closed by Claim 4.3 since $\mathrm{Im}(Kl_k)$ is closed. Note that

$$Cr_{n-k}^* \circ Kl_k^* = (Kl_k \circ Cr_{n-k})^* = T_{n-k,k}^* = T_{k,n-k}$$

implying $B \circ Cr_k = T_{k,n-k}$.

Definition 4.6. A generalized Crofton measure for $\phi \in Val_k^{ev,-\infty}(V)$ is any $\mu \in \Gamma^{-\infty}(E^{n,n-k})$ s.t. $Cr_k(\mu) = \phi$. We proved that such ϕ exists.

4.3 Reconstructing a continuous valuation from its generalized Crofton measure

Lemma 4.7. *Let W be a linear space, $\phi \in Val_k^{ev}(W)$ a continuous valuation, and $\mu_\phi \in \Gamma^{-\infty}(E^{n,k})$ a generalized Crofton measure for ϕ. Let K be a convex body such that $|Pr_{W/\Lambda}(K)| \in \Gamma^\infty(K^{n,n-k}) \otimes D(W)^*$. Then*

$$\phi(K) = \int_{Gr(n,n-k)} |Pr_{W/\Lambda}(K)|\mu_\phi(\Lambda)$$

Proof. A convex body $K \subset W$ is naturally an element of $Val_k^{ev,\infty}(W)^* = Val_{n-k}^{ev,-\infty}(W) \otimes D(W)^*$; denote the corresponding element by $\psi_{K,n-k}$. Then $\psi_{K,n-k} = Kl^*(\gamma_{K,n-k}) = (Cr\otimes Id)(\gamma_{K,n-k})$ for some $\gamma_{K,n-k} \in \Gamma^{-\infty}(E^{n,n-k})\otimes D(W)^*$, and so

$$Cr^*(\psi_{K,n-k}) = (Kl_{n-k}\otimes Id)(\psi_{K,n-k}) = (T_{k,n-k}\otimes Id)(\gamma_{K,n-k}) \in \Gamma^{-\infty}(K^{n,n-k})\otimes D(W)^*$$

In particular, $Cr^*(\psi_{K,n-k})$ lies in the image of the cosine transform.
Let us verify that $Cr^*(\psi_{K,n-k})$ is continuous and $Cr^*(\psi_{K,n-k})(\Lambda) = |Pr_{W/\Lambda}(K)| \in \Gamma(K^{n,n-k}) \otimes D(W)^*$, where $\Lambda \in Gr(V, n-k)$.
Take any smooth Crofton measure $\gamma \in \Gamma^\infty(E^{n,k})$. Then

$$\langle Cr^*(\psi_{K,n-k}), \gamma\rangle = \langle \psi_{K,n-k}, Cr(\gamma)\rangle = Cr(\gamma)(K) = \int_{Gr(n,n-k)} |Pr_{W/\Lambda}(K)|\gamma$$

that is, $Cr^*(\psi_{K,n-k}) = |Pr_{W/\Lambda}(K)|$, so $|Pr_{W/\Lambda}(K)| \in T_{k,n-k}(\Gamma^{-\infty}(E^{n,n-k}))\otimes D(W)^*$. By Proposition 4.2, it follows that $|Pr_{W/\Lambda}(K)| = T_{k,n-k}(\sigma)$ for some $\sigma \in \Gamma^\infty(E^{n,n-k}) \otimes D(W)^*$.

Now fix some Euclidean structure on W. We know that $T_{n-k,k}(\mu_\phi) = Kl(\phi)$. Choose a sequence $\phi_j \in Val_k^{ev,\infty}(W)$ s.t. $\phi_j \to \phi$ in $Val_k^{ev}(W)$, so $\phi_j(K) \to \phi(K)$. Choose Crofton measures $\mu_n \in \Gamma^\infty(E^{n,k})$ s.t. $T_{k,n-k}(\mu_j) = Kl(\phi_j)$. Then since $T_{k,n-k}^* = T_{n-k,k}$,

$$\phi_j(K) = \int_{Gr(n,n-k)} |Pr_{\Lambda^\perp}(K)|\mu_j(\Lambda) = \int_{Gr(n,k)} \sigma T_{n-k,k}(\mu_j) =$$

$$= \int_{Gr(n,k)} \sigma Kl(\phi_j) \to \int_{Gr(n,k)} \sigma Kl(\phi) = \int_{Gr(n,k)} \sigma T_{n-k,k}(\mu_\phi)$$

and since σ is smooth and $T^*_{k,n-k} = T_{n-k,k}$, this equals

$$\int_{Gr(n,n-k)} |Pr_{\Lambda^\perp}(K)|\mu_\phi(\Lambda)$$

as claimed. Q.E.D.

Thus, given a generalized section $s \in \Gamma^{-\infty}(E^{n,k})^{SO^+(n-1,1)}$, we may consider $\phi = Cr_{n-k}(s)$ which is an even, k-homogeneous, Lorentz-invariant generalized valuation. Then one may ask whether a continuous extension to all convex bodies of ϕ exists. According to the Lemma, its value (as a continuous valuations) on all convex bodies with smooth k-support function should be given by the formula

$$\phi(K) = \int_{\Lambda \in Gr(V,n-k)} s(Pr_{V/\Lambda}(K))$$

4.4 Finding the invariant generalized sections

Let X be a smooth manifold, and $Y \subset X$ a smooth compact submanifold. Let E be a smooth vector bundle over Y. Define the sheaf

$$J^q_Y = \{f \in C^\infty(X) : \ L_{X_1}...L_{X_q} f\Big|_Y = 0 \ \forall X_j \in \Gamma^\infty(TX), \ j = 1, ..., q\}$$

Then define $\mathcal{M}^q_Y = J^q_Y \mathcal{M}^\infty(X)$ and

$$\Gamma^{-\infty,q}_Y(E) = \{\phi \in \Gamma^{-\infty}(E) : \forall s \in \Gamma^\infty(E^*), m \in \Gamma(\mathcal{M}^q_Y) \ \phi(s \otimes m) = 0\}$$

Let F^q denote the vector bundle over Y with fiber

$$F^q|_x = Sym^q(N_xY) \otimes D^*(N_xY) \otimes E|_x$$

where $N_xY = T_xX/T_xY$ is the normal space to Y at x. Then $\Gamma^{-\infty,q}_Y(E)/\Gamma^{-\infty,q-1}_Y(E) = \Gamma^{-\infty}(Y, F^q)$.

We thus have a useful tool for finding the G-invariant generalized sections of a vector bundle:

Proposition 4.8. *Let G be a group, X a manifold equipped with G-action, E over X a G-equivariant line bundle, and $Y \subset X$ a compact orbit of G. Then there is an injective map*

$$p : \left(\Gamma^{-\infty,q}_Y(E)/\Gamma^{-\infty,q-1}_Y(E)\right)^G \to \Gamma^\infty(Y, F^q)^G$$

Proof. Taking the G-invariant elements of a G-module is left exact. Therefore, the exact sequence

$$0 \to \Gamma^{-\infty,q-1}_Y(E) \to \Gamma^{-\infty,q}_Y(E) \to \Gamma^{-\infty}(Y, F^q) \to 0$$

gives an injection

$$\left(\Gamma^{-\infty,q}_Y(E)/\Gamma^{-\infty,q-1}_Y(E)\right)^G \to \Gamma^{-\infty}(Y, F^q)^G$$

So it remains to verify that in fact $\Gamma^{-\infty}(Y, F^q)^G \subset \Gamma^\infty(Y, F^q)$. This holds because G acts transitively on Y: we can choose any smooth probability measure with compact support μon G , and then $\forall f \in \Gamma^{-\infty}(Y, F^q)^G, \ f = f * \mu \in \Gamma^\infty(Y, F^q)$. Q.E.D.

4.4.1 Construction of some generalized functions on the unit circle

For the following, define $c_j(\lambda)$ by

$$\left(\frac{\sin x}{x}\right)^\lambda = \sum_{j=0}^\infty c_j(\lambda) x^{2j}$$

The series converge locally uniformly in $x \in (-\pi,\pi)$ for every $\lambda \in \mathbb{C}$, in particular $\sum_{j=0}^\infty |c_j(\lambda)|$ converges. The coefficients $c_j(\lambda)$ are polynomial functions of $\lambda \in \mathbb{C}$: $c_0(\lambda) = 1$, $c_1(\lambda) = -\frac{\lambda}{3!}$, $c_2(\lambda) = \frac{\lambda}{5!} + \frac{\lambda(\lambda-1)}{2\cdot 3!^2}$, $c_3(\lambda) = -\frac{\lambda}{7!} - \frac{\lambda(\lambda-1)}{3!5!} + \frac{\lambda(\lambda-1)(\lambda-2)}{6\cdot 3!^3}$ and so on.

Lemma 4.9. *For every $k \in \mathbb{Z}$, the function $I_k(\lambda) : \mathbb{C} \to \mathbb{C}$ given by*

$$I_k(\lambda) = \int_0^1 x^k |\sin x|^\lambda dx$$

for $Re\lambda > 0$, admits a meromorphic extension to the complex plane, with simple poles at $\lambda = -(k+2j+1)$, $j = 0,1,2,...$ and residues $Res(I_k, -k-2j-1) = 2c_j(-k-2j-1)$.

Proof. Write

$$I_k(\lambda) = \int_0^1 x^{k+\lambda} \left(\frac{\sin x}{x}\right)^\lambda dx =$$

$$= \sum_{j=0}^\infty c_j(\lambda) \frac{1}{\lambda+k+2j+1}$$

is meromorphic with simple poles at $\lambda = -k-2j-1$, $j \geq 0$. Q.E.D.

Lemma 4.10. *There exists a meromorphic map $\sin_+^\lambda x : \mathbb{C} \to C^{-\infty}(-\pi,\pi)$ with simple poles at $\lambda = -1,-2,...$ and residues*

$$Res(\sin_+^\lambda, -k) = \begin{cases} \sum_{j=0}^m \frac{1}{(2j)!} c_{m-j}(-k)\delta_0^{(2j)}, k = 2m+1 \\ -\sum_{j=0}^{m-1} \frac{1}{(2j+1)!} c_{m-1-j}(-k)\delta_0^{(2j+1)}, k = 2m \end{cases}$$

s.t. for all $\lambda \notin \mathbb{Z}_{<0}$, $\sin_+^\lambda x(\phi dx) = \int_0^\pi \phi(x)\sin^\lambda x dx$ for $\phi \in C_c^\infty(-\pi,\pi)$ that vanishes in a neighborhood of 0.

Proof. For $Re(\lambda) > -1$, $\sin_+^\lambda x$ is locally integrable near 0 and so $\sin_+^\lambda x \in C^{-\infty}(-1,1)$ is well-defined and analytic in λ. A meromorphic continuation with the desired properties in the region $Re(\lambda) > -(k+1)$ is given for $\phi \in C_c(-\pi,\pi)$ by

$$\sin_+^\lambda x(\phi dx) = \int_1^\pi \phi(x)\sin^\lambda x dx + \int_0^1 \sin^\lambda x(\phi(x) - \phi(0) - x\phi'(0) - ... - \frac{1}{(k-1)!}x^{k-1}\phi^{(k-1)}(0))dx +$$

$$+\phi(0)I_0(\lambda) + \phi'(0)I_1(\lambda) + ... + \frac{1}{(k-1)!}\phi^{(k-1)}(0)I_{k-1}(\lambda)$$

by the Lemma above, this is a well-defined generalized function, meromorphic in λ, with simple poles at $\lambda = -1,-2,...$ and residues as claimed. Q.E.D.

We define also $\sin_-^\lambda x \in C^{-\infty}(-\pi,\pi)$ by $\langle \sin_-^\lambda x, \phi(x)dx\rangle = \langle \sin_+^\lambda x, \phi(-x)dx\rangle$. Then

$$\text{Res}(\sin_-^\lambda x, -k) = \begin{cases} \sum_{j=0}^m \frac{1}{(2j)!} c_{m-j}(-k)\delta^{(2j)}, k = 2m+1 \\ \sum_{j=0}^{m-1} \frac{1}{(2j+1)!} c_{m-1-j}(-k)\delta^{(2j+1)}, k = 2m \end{cases}$$

Before formulating the main result of this subsection, recall the following

Claim. Let $f : \mathbb{C} \to C^{-\infty}(X)$, $\lambda \mapsto f_\lambda(x)$ be meromorphic, where X is a smooth manifold. Assume that λ_0 is a simple pole, and $h(x) \in C(X)$ positive s.t. $f_\lambda(gx) = h(x)^\lambda f_\lambda(x)$ in the holomorphic domain of f_λ, for some $g \in \mathrm{Diff}(X)$. Then $r(x) = \mathrm{Res}(f_\lambda; \lambda_0)$ satisfies the same equation.

Proof. Indeed, write $f_\lambda(x) = \frac{a_{-1}(x)}{\lambda-\lambda_0} + a_0(x) + \ldots$, so that $r(x) = a_{-1}(x)$. Then

$$f_\lambda(gx) = h(x)^\lambda f_\lambda(x) \Rightarrow \frac{a_{-1}(gx)}{\lambda - \lambda_0} + a_0(gx) + \ldots = \frac{a_{-1}(x)h(x)^\lambda}{\lambda - \lambda_0} + a_0(x)h(x)^\lambda + \ldots$$

Developing $h(x)^\lambda$ into power series near $\lambda = \lambda_0$ we see that

$$a_{-1}(gx) = a_{-1}(x)h(x)^{\lambda_0}$$

as claimed.

Recall the Lorentz form Q on $\mathbb{R}^2$, which we now restrict to the unit circle S^1. Then $\{Q \geq 0\} = \{-\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{4}\} \cup \{\frac{3\pi}{4} \leq \alpha \leq \frac{5\pi}{4}\}$ and $\{Q \leq 0\} = \{\frac{\pi}{4} \leq \alpha \leq \frac{3\pi}{4}\} \cup \{\frac{5\pi}{4} \leq \alpha \leq \frac{7\pi}{4}\}$. Q.E.D.

Corollary 4.11. *(a)For any sign $\epsilon \in \{+,-\}$, there is a meromorphic in λ, generalized function f^ϵ_λ on S^1, namely $\cos^\lambda_\epsilon(2\alpha)$ (here α is the angle on the circle) with simple poles at $\lambda = -1, -2, \ldots$ that is supported on sign$Q \in \{0\} \cup \{\epsilon\}$, which satisfies for every $\phi \in C^\infty(S^1)$ vanishing in a neighborhood of the light cone*

$$\langle f^\epsilon_\lambda, \phi(\alpha)d\alpha\rangle = \int_{signQ(\alpha)=\varepsilon} |\cos 2\alpha|^\lambda \phi(\alpha)d\alpha$$

and

$$(g^{-1})^*(f_\lambda)(t) = \kappa^\lambda \Big(\frac{1+\kappa^2 t^2}{1+t^2}\Big)^{-\lambda} f_\lambda(t) \qquad (5)$$

for $g = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$, *where* $g_*(f_\lambda) = f_\lambda \circ g$, $\kappa = e^{-2\theta}$, $t = \tan(\frac{\pi}{4} - \alpha)$.
(b) For $\lambda = -k$, $k = 1, 2, \ldots$ the residue

$$Res(f^\epsilon_\lambda; -k) =$$

$$= \begin{cases} \sum_{j=0}^{m} \frac{1}{(2j)!2^{2j}} c_{m-j}(-k)(\delta^{(2j)}_{\alpha=\pi/4} + \delta^{(2j)}_{\alpha=5\pi/4} - \delta^{(2j)}_{\alpha=3\pi/4} - \delta^{(2j)}_{\alpha=7\pi/4}),\ k = 2m+1 \\ -\epsilon\sum_{j=0}^{m-1} \frac{1}{(2j+1)!2^{2j+1}} c_{m-1-j}(-k)(\delta^{(2j+1)}_{\alpha=\pi/4} + \delta^{(2j+1)}_{\alpha=5\pi/4} - \delta^{(2j+1)}_{\alpha=3\pi/4} - \delta^{(2j+1)}_{\alpha=7\pi/4}),\ k = 2m \end{cases}$$

satisfies equation 5. Also, the linear combination

$$f^+_\lambda + (-1)^k f^-_\lambda$$

is holomorphic at $\lambda = -k$ and satisfies equation 5.

Proof. (a) This can be verified directly for $\mathrm{Re}\lambda > 0$, similarly to equation 7. Then, both sides of the equation are memoromphic maps $\mathbb{C} \to C^{-\infty}(S^1)$ so uniqueness of meromorphic extension applies. For statement (b) concerning residues (the second half is immediate from (a)), we use the Claim above. Q.E.D.

Remark 4.12. All the generalized functions on S^1 that we defined are even, and so define generalized functions on $\mathbb{RP}^1$. Let Q denote the Lorentz quadratic form on $\mathbb{R}^2$. The Q-orthogonal complement of a line in $\mathbb{R}^2$ (which is the same as reflection w.r.t. to the light cone) induces a $\mathbb{Z}_2$-action on $\mathbb{RP}^1$ and so also on $C^{-\infty}(\mathbb{RP}^1)$. We call $f \in C^{-\infty}(\mathbb{RP}^1)$ cone-symmetric or cone-antisymmetric according to the action of $\mathbb{Z}_2$ on it. Then for $\lambda \neq -k$, $\cos^\lambda_+(2\alpha) + \cos^\lambda_-(2\alpha)$ is cone-symmetric and $\cos^\lambda_+(2\alpha) - \cos^\lambda_-(2\alpha)$ is cone-antisymmetric; for $\lambda = -k$, there are two cases:

- k is odd, then $\mathrm{Res}(\cos_\pm^\lambda(2\alpha), -k)$ is cone-symmetric and $\cos_+^\lambda(2\alpha) - \cos_-^\lambda(2\alpha)$ is cone-antisymmetric.
- k is even, then $\mathrm{Res}(\cos_\pm^\lambda(2\alpha), -k)$ is cone-antisymmetric and $\cos_+^\lambda(2\alpha) + \cos_-^\lambda(2\alpha)$ is cone-symmetric.

We will denote the cone-symmetric and cone-antisymmetric functions corresponding to λ by $f_\lambda^+(\alpha)$ and $f_\lambda^-(\alpha)$, respectively, normalized so that $f^+_{-(2j+1)} = \mathrm{Res}(f_\lambda^+, -(2j+1))$ and $f^-_{-2j} = \mathrm{Res}(f_\lambda^-, -2j)$. Note that $f_\lambda^\pm$ is invariant to reflection w.r.t. the origin and to both coordinate axes.
For non-integer λ, we write f_λ^T and f_λ^S for the functions corresponding to $\cos_-^\lambda(2\alpha)$ and $\cos_+^\lambda(2\alpha)$, resp. (standing for the time-like and space-like support of the function).

Remark 4.13. Note that the generalized functions supported on the light cone correspond to the residues, and they are given by derivatives of order $k-1$ for $\lambda = -k$ since $c_0(\lambda) \equiv 1$.

We will now construct generalized functions $f^\pm_{n,k,\lambda} \in C^{-\infty}(Gr(n,k))$ that are $SO(n-1)$-invariant, have singular support on the light cone, and satisfy the following transformation law under the Lorentz group: Fix any $(k-1)$-dimensional $\tilde{\Lambda} \subset \mathbb{R}^{n-1}$ (the space coordinate plane), and $v \in \mathbb{R}^{n-1}$ orthogonal to $\tilde{\Lambda}$. Denote $\Pi = \mathrm{Span}\{v, e_n\}$. Let $g \in G$ be a θ-boost in Π, namely

$$g = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

and extended by identity in the orthogonal direction. Denote

$$\Lambda_\alpha = \tilde{\Lambda} + R_\alpha v$$

where R_α denotes rotation by α in Π, extended by the identity in the orthogonal directions. Then

$$(g^{-1})^*(f^\pm_{n,k,\lambda})(\Lambda_\alpha) = \kappa^\lambda \left(\frac{1+\kappa^2 t^2}{1+t^2}\right)^{-\lambda} f^\pm_{n,k,\lambda}(\Lambda_\alpha) \tag{6}$$

where $\langle g_*(f_\lambda), \mu\rangle = \langle f_\lambda, (g^{-1})_*\mu\rangle$, $\kappa = e^{-2\theta}$, $t = \tan(\frac{\pi}{4} - \alpha)$.
Here and in the following, $\alpha : Gr(n,k) \to [0, \frac{\pi}{2}]$ is the elevation angle of $\Lambda \in Gr(n,k)$ above the space coordinate hyperplane.

This is achieved as follows: choose a smooth function $\chi \in C^\infty(S^1)$ invariant to reflection w.r.t both coordinate axes, s.t. χ vanishes in a 2ϵ-neighborhood of the poles and of the equator, and equals 1 outside a 3ϵ-neighborhood of the poles and equator. Let $f \in C^{-\infty}(S^1)$ be any generalized function smooth near the poles and the equator, and invariant to reflections w.r.t. both axes.

Define $C_\epsilon = \{\Lambda \in Gr(n,k) : \alpha(\Lambda) \geq \frac{\pi}{2} - \epsilon\}$ and $E_\epsilon = \{\Lambda \in Gr(n,k) : \alpha(\Lambda) \leq \epsilon\}$. Outside $C_\epsilon \cup E_\epsilon$, one has the well-defined smooth submersion $\alpha : S^{n-1} \setminus (C_\epsilon \cup E_\epsilon) \to (\epsilon, \frac{\pi}{2} - \epsilon)$. So we may pull-back χf as follows: define $u = \alpha^*(\chi f) \in C^{-\infty}(Gr(n,k))^{SO(n-1)}$ (which we extend to $C_\epsilon \cup E_\epsilon$ by zero).
Now observe that α^2 is a smooth function on $E_{3\epsilon}$: this can be seen by writing

$$\sin^2\alpha = \sum_{j=1}^k \langle v_j, e_n\rangle^2$$

where $\{v_j\}$ is any orthonormal basis of Λ, and e_n the unit vector in the time direction. Also, $(\frac{\pi}{2}-\alpha)^2$ is smooth in $C_{3\epsilon}$. Since the function $(1-\chi)f \in C^\infty(S^1)$ is smooth and invariant to reflections w.r.t. both coordinate axes, by Lemma 3.4 (applied separately near $\alpha = 0$ and $\alpha = \frac{\pi}{2}$) one may define a smooth $SO(n-1)$-invariant function $v(\Lambda) = \Big((1-\chi)f\Big)(\alpha(\Lambda)) \in C^\infty(Gr(n,k))^{SO(n-1)}$ supported in $C_{3\epsilon} \cup E_{3\epsilon}$. Now define $Gr_{n,k}(f) = u + v \in C^{-\infty}(Gr(n,k))^{SO(n-1)}$.
We now define $f^{\pm}_{n,k,\lambda} = Gr_{n,k}(f^{\pm}_{\lambda})$ for non-integer λ. Then for values of λ satisfying $\mathrm{Re}\lambda > 0$, verifying that $f^{\pm}_{n,k,\lambda}$ satisfies equation 6 amounts to a testing the numerical equation given by 5. As before for S^1, we conclude by meromorphic extension that the equation is satisfied for all values of λ that are not odd resp. even negative integers for $f^{+}_{n,k,\lambda}$ resp. $f^{-}_{n,k,\lambda}$. Finally we define $f^{-}_{n,k,-2j}$ and $f^{+}_{n,k,-(2j+1)}$ by taking the respective residues.

Let us write an explicit formula for $f^{\pm}_{n,k,\lambda}(\mu)$ for $\mu \in \mathcal{M}^\infty(Gr(n,k))^{SO(n-1)}$. Writing $\mu = \phi(\alpha)d\Lambda$ where $d\Lambda$ is the unique $SO(n)$-invariant probability measure on $Gr(n,k)$ we claim that

$$f^{\pm}_{n,k,\lambda}(\mu) = f^{\pm}_{\lambda}(\phi(\alpha)g_{n,k}(\alpha)d\alpha)$$

with $g_{n,k}(\alpha) = C_{n,k}\cos^{n-k-1}\alpha\sin^{k-1}\alpha$.
Indeed, by uniqueness of meromorphic continuation it is enough to verify the formula for $\mathrm{Re}\lambda > 0$. Then $f^{\pm}_{\lambda}$ is continuous and $f^{\pm}_{n,k,\lambda}(\Lambda) = f^{\pm}_{\lambda}(\alpha(\Lambda))$. So we may write

$$f^{\pm}_{n,k,\lambda}(\mu) = \int_{Gr(n,k)} f^{\pm}_{\lambda}(\alpha(\Lambda))\phi(\alpha(\Lambda))d\Lambda$$

and integrate along submanifolds of constant elevation. It remains to see that $\alpha_*(d\Lambda) = g_{n,k}(\alpha)d\alpha$. The angle $\beta = \frac{\pi}{2} - \alpha$ betwen a random (w.r.t. the Haar measure on $Gr(n,k)$) k-dimensional subspace and a fixed direction is distributed as the angle between a random vector $x \in S^{n-1}$ (w.r.t. the Haar measure) and a fixed k-subspace. Since $\{x \in S^{n-1} : \angle(v, \mathbb{R}^k) = \beta\} = \{x \in S^{n-1} : x_1^2 + ... + x_k^2 = \cos^2\beta\} = \Big(\cos\beta S^{k-1}\Big) \times \Big(\sin\beta S^{n-k-1}\Big)$,

$$g_{n,k}(\alpha) = C_{n,k}\cos^{k-1}\beta\sin^{n-k-1}\beta = C_{n,k}\cos^{n-k-1}\alpha\sin^{k-1}\alpha$$

4.4.2 The case $k = 1$

We will denote $X = Gr(V, 1)$, $M \subset X$ will be the set of Q- degenerate subspaces, referred to as the light cone in X. We denote by α the angle between a line $\Lambda \in X$ and the space coordinate hyperplane. We start by proving

Proposition 4.14. *The G-invariant generalized sections of $K^{n,1}$ are spanned by $|\cos 2\alpha|^{\frac{1}{2}}s_0$ and $sign(\cos 2\alpha)|\cos 2\alpha|^{\frac{1}{2}}s_0$, where s_0 is the Euclidean section.*

Proof. We should only prove that there are no sections supported on the light cone, denoted M. Assume $f \in \Gamma^{-\infty}(X, K^{n,1})$ is supported on the light cone and G-invariant.
In our case, the action of G on $X = Gr(n,1)$ is given by

$$\tan\beta = \frac{\tan\alpha + \tanh\theta}{1 + \tan\alpha\tanh\theta}$$

where $g = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$ and $\beta = g\alpha$. In particular

$$d\beta = \frac{d\alpha}{\cosh 2\theta + \sin 2\alpha\sinh 2\theta}$$

The action of G on the fibers is given by

$$g_*(\phi s_0)(\beta) = \phi(\alpha)\frac{|\cos 2\beta|^{\frac{1}{2}}}{|\cos 2\alpha|^{\frac{1}{2}}}s_0(\beta)$$

(with the value at $\alpha = \beta = \frac{\pi}{4}$ understood in the limit sense). We change the coordinates as follows: $\epsilon = \frac{\pi}{4} - \alpha$, $\eta = \frac{\pi}{4} - \beta$ and $t = \tan\epsilon$, $s = \tan\eta$. Also, denote

$$\kappa = \frac{1-\tanh\theta}{1+\tanh\theta} = \frac{1}{(\cosh\theta + \sinh\theta)^2} = e^{-2\theta}$$

This corresponds to

$$s = \kappa t$$

and

$$g(\phi s_0)(s) = \phi(t)\kappa^{\frac{1}{2}}\left(\frac{1+\kappa^2 t^2}{1+t^2}\right)^{-\frac{1}{2}}s_0(s) \tag{7}$$

Now the existence for some $q \geq 0$ of an invariant generalized section supported on M (corresponding to $t_0 = 0$) would imply according to 4.8 the existence of a non-zero invariant section over M of $F = \Gamma_M^{-\infty,q}(X, K^{n,1})/\Gamma_M^{-\infty,q-1}(X, K^{n,1}) = D^*(NM) \otimes Sym^q(NM) \otimes K^{n,1}|_M = D^*(NM) \otimes (NM)^{\otimes q} \otimes K^{n,1}|_M$ (for the last equality note that NM is a line bundle).
Note that for $l \in M$, $N_l M = T_l X/T_l M = (l^* \otimes (V/l))/(l^* \otimes (l^Q/l)) \backsimeq l^* \otimes (V/l^Q)$, where l^Q is the Q-orthogonal complement of l, and $l \in M \iff l \subset l^Q$.
Applying a pseudo-rotation (boost) by pseudo-angle θ fixing l, the resulting transformation of the fiber of $F|_l$ is multiplication by

$$\kappa \cdot \kappa^q \cdot \kappa^{1/2}$$

for $\kappa = e^{-2\theta}$, which cannot equal 1 for any q. We conclude there are no invariant sections supported on the light cone. Q.E.D.

When $k = 1$, the Crofton fiber $E^{n,1}|_\Lambda$ is canonically isomorphic (in particular, as G-equivariant bundles) to $D(V)^n \otimes D(\Lambda)^{*(n+1)}$:

$$D(V/\Lambda) \otimes D(T_\Lambda Gr(n, n-1)) = D(V/\Lambda) \otimes |\wedge^{top}((V/\Lambda)^* \otimes \Lambda)| =$$
$$= D(V/\Lambda) \otimes D((V/\Lambda)^{\otimes(n-1)}) \otimes |\Lambda^{\wedge top}| = D(V/\Lambda)^n \otimes D(\Lambda)^* =$$
$$= D(V)^n \otimes D(\Lambda)^{*(n+1)}$$

Let α be the anglular altitude on the sphere, and z_0 be the Euclidean section of the bundle $E^{n,1}$. The transformation rule under the G-action for a boost g_θ by pseudo-angle θ is therefore

$$g_*(\phi z_0)(\beta) = \phi(\alpha)\frac{|\cos 2\beta|^{-\frac{n+1}{2}}}{|\cos 2\alpha|^{-\frac{n+1}{2}}}s_0(\beta)$$

or equivalently

$$g(\phi z_0)(s) = \phi(t)\kappa^{-\frac{n+1}{2}}\left(\frac{1+\kappa^2 t^2}{1+t^2}\right)^{\frac{n+1}{2}}z_0(s) \tag{8}$$

where $t = \tan(\frac{\pi}{4} - \alpha)$, $s = \tan(\frac{\pi}{4} - \beta)$, $\beta = g_\theta\alpha$, $s = \kappa t$, and $\kappa = e^{-2\theta}$

Let f be a G-invariant generalized section of this bundle. When restricted to an open orbit, such a section must be smooth (since an open orbit is a homogeneous manifold for G). Therefore, on the open orbits $f = C|\cos 2\alpha|^{-\frac{n+1}{2}}z_0$, C a locally constant function on $Gr(V, n-1)$.

In light of Corollary 4.1, we get

Corollary 4.15. *The space* $\Gamma^{-\infty}(Gr(n,n-1),E^{n,1})^G$ *is at most 2-dimensional*

We will now turn to constructing two independent sections of this space, proving is it in fact 2-dimensional. Let us first remark that applying Proposition 4.8 for this manifold (this time $T_\Lambda M=(\Lambda/\Lambda^Q)^*\otimes(V/\Lambda)$), one can see that an invariant generalized section supported on the light cone can exist only if

$$q+1-\frac{n+1}{2}=0 \iff n=2q+1$$

where q is the order of the section (as a differential operator). We will show that such sections do indeed exist.

Proposition 4.16. $\dim\Gamma^{-\infty}(E^{n,1})^G=2$. *For odd* n*, there is a one dimensional subspace of generalized sections supported on the light cone. For even* n*, none are supported on the light cone.*

Proof. The sections are associated with the generalized functions on $Gr(n,n-1)$ constructed in 4.4.1.
According to equations 6 and 8, they are given (after a Euclidean trivialization) by $f^{\pm}_{n,n-1,\lambda}$ with $\lambda=-\frac{n+1}{2}$. The support properties follow immediately from the corresponding properties for $f^{\pm}_\lambda$. Q.E.D.

Those sections will be denoted $f^{\pm}_{n,1}$.
Let us write explicit formulas for those sections in some dimensions: For $n=3$, the cone-symmetric section $f^+_{3,1}$ (after rescaling) is given by

$$\phi(\alpha,\psi)d\sigma\mapsto\int_{\epsilon=0}^{\frac{\pi}{4}}\int_{\psi=0}^{2\pi}\frac{\phi(\frac{\pi}{4}+\epsilon,\psi)+\phi(\frac{\pi}{4}-\epsilon,\psi)-2\phi(\frac{\pi}{4},\psi)}{|\sin 2\epsilon|^2}\sin(\epsilon+\frac{\pi}{4})d\epsilon d\psi+$$

$$+\sqrt{2}I_0(-2)\int_{\psi=0}^{2\pi}\phi(\frac{\pi}{4},\psi)d\psi$$

and the cone-antisymmetric section $f^-_{3,1}$ is given by

$$\phi(\alpha,\psi)d\sigma\mapsto\frac{\partial}{\partial\alpha}\bigg|_{\alpha=\frac{\pi}{4}}(\sin\alpha\int_{S^1}d\psi\phi(\alpha,\psi))$$

For higher odd values of n, the cone-antisymmetric section is given by

$$\phi(\alpha,\psi)d\sigma\mapsto\frac{\partial^m}{\partial\alpha^m}\bigg|_{\alpha=\frac{\pi}{4}}(\sin^{n-2}\alpha\int_M\phi(\alpha,\psi)d\psi)+\text{lower order derivatives}$$

where $m=\frac{n-1}{2}$.

4.4.3 Case of general k

Denote $X=Gr(V,k)$, M the set of Q-degenerate subspaces.

Proposition 4.17. *There are no* G*-invariant sections over* M *of the bundle with fiber over* Λ *equal to* $D^*(N_\Lambda M)\otimes Sym^q(N_\Lambda M)\otimes K^{n,k}|_\Lambda$.

Proof. Fix $\Lambda\in M$ touching the light cone C along the line $l=\Lambda\cap\Lambda^Q$. Denote also $\Omega=\Lambda+\Lambda^Q=l^Q$. Write $N_\Lambda M=T_\Lambda X/T_\Lambda M$. Then

$$N_\Lambda M=l^*\otimes(V/\Omega)$$

Thus as in the case $k=1$, for $g=g_\theta\in\mathrm{Stab}(\Lambda)$, the action on $D^*(N_\Lambda M)\otimes Sym^q(N_\Lambda M)\otimes K^{n,k}|_\Lambda$ is by multiplication by $\kappa^{q+1}\kappa^{1/2}$ where $\kappa=e^{-2\theta}$. So again by Proposition 4.8, there are no invariant generalized sections of $K^{n,k}$ supported on the light cone. Q.E.D.

Therefore by Proposition 2.3, $\dim \Gamma^{-\infty}(X, K^{n,k})^G = 2$.

Proposition 4.18. *$\dim \Gamma^{-\infty}(E^{n,k})^G = 2$ for all $1 \le k \le n-1$. For odd n, there is a one dimensional subspace of generalized sections supported on the light cone. For even n, none are supported on the light cone.*

Proof. Again by Corollary 4.1, $\dim \Gamma^{-\infty}(E^{n,k})^G \le 2$. Let us find two independent sections explicitly. This time $\Lambda \in Gr(V, n-k)$ and

$$\begin{aligned} E_\Lambda = D(V/\Lambda) \otimes D(T_\Lambda Gr(V, n-k)) = \\ = D(V) \otimes D(\Lambda)^* \otimes D(\Lambda^* \otimes V/\Lambda) = \\ = D(V) \otimes D^*(\Lambda) \otimes D(V)^{n-k} \otimes D^*(\Lambda)^n = \\ = D(V)^{n-k+1} \otimes D(\Lambda)^{*(n+1)} \end{aligned}$$

So similarly to the case $k = 1$, the invariant sections $f^{\pm}_{n,k}$ of $E^{n,k}$ are given, after the Euclidean trivialization, by $f^{\pm}_{n,n-k,\lambda}$, with $\lambda = -\frac{n+1}{2}$. Q.E.D.

For even values of n, we will also use the basis $f^S_{n,k}, f^T_{n,k}$ corresponding to f^S_λ, f^T_λ.

Recall that for $\mu \in \mathcal{M}^\infty(Gr(n,k))^{SO(n-1)}$ such that $\mu = \phi(\alpha) d\Lambda$ where $d\Lambda$ is the unique $SO(n)$-invariant probability measure on $Gr(n, n-k)$ we have

$$f^{\pm}_{n,k}(\mu) = f^{\pm}_\lambda(\phi(\alpha) g_{n,n-k}(\alpha) d\alpha)$$

where $g_{n,n-k}(\alpha) = C_{n,k} \sin^{n-k-1}\alpha \cos^{k-1}\alpha$ and $\lambda = -\frac{n+1}{2}$. From now on, we renormalize $f^{\pm}_{n,k}$ so that $C_{n,k} = 1$.

Theorem 4.19. *For all $1 \le k \le n-1$, $\dim Val_k^{ev,-\infty}(\mathbb{R}^n)^{SO^+(n-1,1)} = 2$.*

Proof. According to Proposition 4.4, $\left(Val_k^{ev,-\infty}(V)\right)^G$ is naturally a subspace of $\left(\Gamma^{-\infty}(K^{n,k})\right)^G$. In particular, $\dim \left(Val_k^{ev,-\infty}(V)\right)^G \le 2$. Then by Proposition 4.5, $\mathrm{Ker}\Big(Cr_{n-k} : \Gamma^{-\infty}(E^{n,k}) \to Val_k^{ev,-\infty}(V)\Big) \subset \mathrm{Ker} T_{n-k,k}$ so by Corollary 4.1 one has $\dim Val_k^{ev,-\infty}(V)^G \ge 2$. Thus, we get equality. Q.E.D.

It follows that every $SO^+(n-1,1)$-invariant continuous valuation $\phi \in Val_k(V)$ is determined by its uniquely-defined $SO^+(n-1,1)$-invariant generalized Crofton measure.

5 The non-existence of even Lorentz-invariant valuations for $1 \le k \le n-2$

We now proceed to show that the generalized valuations $\phi = Cr(f^{\pm}_{n,k})$ corresponding to the sections $f^{\pm}_{n,k} \in \Gamma^{-\infty}(E^{n,k})^G$ that we found, are not continuous valuations. In fact, we will show those valuations cannot be extended by continuity to the double cone. By Lemma 4.7, it follows that for an $SO(n-1)$-invariant smooth unconditional body K^n with k-support function $h_k(\alpha; K)$, those valuations are given by

$$\phi(K^n) = f^{\pm}_\lambda(h_k(\alpha; K) g_{n,n-k}(\alpha) d\alpha)$$

with $\lambda = -\frac{n+1}{2}$. Then by 3.7, the same formula holds as long as $h_k(\alpha; K)$ is smooth near the light cone.

5.1 Computations related to the double cone

In the following, $C \subset \mathbb{R}^2$ is the unit ball of the l_1 norm. We will write $h_k(\alpha) = h_k(\alpha; C^{k+1})$ (where $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ is the angle between the normal to the hyperplane to which C^{k+1} is projected, and the space-like coordinate hyperplane). It can be computed as follows: fix $u = (\cos\alpha, 0, ..., 0, \sin\alpha)$ the normal to the hyperplane, and $v = (\cos\beta w, \sin\beta)$, $w \in S^{k-1}$. The surface area measure of C^{k+1} is $\sigma_{C^{k+1}}(v) = \delta_{\frac{\pi}{4}}(\beta) + \delta_{-\frac{\pi}{4}}(\beta)$ and

$$h_k(\alpha) = T_k(\sigma_{C^{k+1}}(\beta))(\alpha) = \int_{S^k} (\delta_{\frac{\pi}{4}}(\beta) + \delta_{-\frac{\pi}{4}}(\beta))|\langle u, v\rangle| \cos^{k-1}\beta d\beta d\sigma_{k-1}(w)$$

If $k > 2$, we take $0 - \frac{\pi}{2} \le \phi \le \frac{\pi}{2}$ to be the elevation angle of $w \in S^{k-1}$. If $k = 2$, $-\pi \le \phi \le \pi$. Let us write $a_k \le \phi \le b_k$ for both cases. Then

$$h_k(\alpha) = C_k \int_{a_k}^{b_k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\delta_{\frac{\pi}{4}}(\beta) + \delta_{-\frac{\pi}{4}}(\beta))|\sin\beta\sin\alpha + \cos\beta\cos\alpha\sin\phi| \cos^{k-1}\beta \cos^{k-2}\phi d\beta d\phi =$$

$$= \frac{C_k}{2^{k/2}} \int_{a_k}^{b_k} (|\sin\alpha + \cos\alpha\sin\phi| + |\sin\alpha - \cos\alpha\sin\phi|)\cos^{k-2}\phi d\phi =$$

$$= \frac{2C_k}{2^{k/2}} \int_{a_k}^{b_k} |\sin\alpha - \cos\alpha\sin\phi| \cos^{k-2}\phi d\phi$$

$$= \begin{cases} h_k^+(\alpha),\ \frac{\pi}{4} \le \alpha \le \frac{\pi}{2} \\ h_k^-(\alpha),\ 0 \le \alpha \le \frac{\pi}{4} \end{cases}$$

Denoting $A_k = \int_{-\pi/2}^{\pi/2} \cos^{k-2}\phi d\phi$, and replacing C_2 by $2C_2$ for $k = 2$, we get

$$h_k^+(\alpha) = \frac{2C_k}{2^{k/2}} A_k \sin\alpha$$

and

$$h_k^-(\alpha) = \frac{2C_k}{2^{k/2}} \left(A_k \sin\alpha - 2\sin\alpha \int_{\arcsin\tan\alpha}^{\pi/2} \cos^{k-2}\phi d\phi + \frac{2}{k-1} \frac{(\cos 2\alpha)^{\frac{k-1}{2}}}{(\cos\alpha)^{k-2}} \right) =$$

$$= h_k^+(\alpha) + \frac{2C_k}{2^{k/2}} \left(\frac{2}{k-1} \frac{(\cos 2\alpha)^{\frac{k-1}{2}}}{(\cos\alpha)^{k-2}} - 2\sin\alpha \int_{\tan\alpha}^{1} (1-t^2)^{\frac{k-3}{2}} dt \right) =$$

with the exception

$$h_1(\alpha) = \frac{1}{\sqrt{2}}(|\sin(\alpha + \frac{\pi}{4})| + |\cos(\alpha + \frac{\pi}{4})|) = \max(|\sin\alpha|, |\cos\alpha|)$$

For $\epsilon > 0$ and every n define the $\epsilon-$ stretching of $\mathbb{R}^n$, S_ϵ to be the diagonal $n \times n$ matrix $c_\epsilon \mathrm{diag}(1, ..., 1, \tan(\frac{\pi}{4} + \epsilon))$ where where $c_\epsilon \to 1$ as $\epsilon \to 0$ will be specified shortly. In the following, we will denote $\eta = \tan(\frac{\pi}{4} + \epsilon)$. We replace the double cone with its $\epsilon-$ stretching $C_{n,\epsilon} = S_\epsilon C^n$, and take c_ϵ such that $h_k(\frac{\pi}{4}; C_\epsilon) = \eta h_k(\frac{\pi}{4}; C)$. We will write in the following $h_{k,\epsilon}(\alpha) = h_k(\alpha; C_{n,\epsilon})$, omitting ϵ when $\epsilon = 0$. Again for all $k \le n - 1$

$$h_{k,\epsilon}(\alpha) = c_\epsilon C_k \int_{a_k}^{b_k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\delta_{\frac{\pi}{4}+\epsilon}(\beta) + \delta_{-\frac{\pi}{4}-\epsilon}(\beta))|\sin\beta\sin\alpha + \cos\beta\cos\alpha\sin\phi| \cos^{k-1}\beta\cos^{k-2}\phi d\beta d\phi$$

Let us write

$$h_{k,\epsilon}(\alpha) = \begin{cases} h_{k,\epsilon}^+(\alpha),\ \alpha \ge \frac{\pi}{4} - \epsilon \\ h_{k,\epsilon}^-(\alpha),\ 0 \le \alpha \le \frac{\pi}{4} - \epsilon \end{cases}$$

where for $k \geq 2$ (again the definition of C_k for $k = 2$ is twice the definition of C_k for $k \geq 3$)

$$h^+_{k,\epsilon}(\alpha) = \frac{2C_k}{2^{k/2}} A_k \eta \cdot \sin\alpha$$

and

$$h^-_{k,\epsilon}(\alpha) = \frac{2C_k}{2^{k/2}}\Bigg(\eta\sin\alpha\Big(A_k - 2\int_{\arcsin(\eta\tan\alpha)}^{\pi/2}\cos^{k-2}\phi d\phi\Big) +$$

$$+\frac{2}{k-1}\cos\alpha(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}\Bigg) =$$

$$= h^+_{k,\epsilon}(\alpha) + \frac{2C_k}{2^{k/2}}\left(\frac{2}{k-1}\cos\alpha(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}} - 2\eta\sin\alpha\int_{\eta\tan\alpha}^{1}(1-t^2)^{\frac{k-3}{2}}dt\right)$$

While

$$h_1(\alpha; C_{n,\epsilon}) = \max(\eta|\sin\alpha|, |\cos\alpha|)$$

By rescaling the bodies, and since we will be only considering a single value of k at a time, we may assume in the subsequent computations that $\frac{2C_k}{2^{k/2}} = 1$ for all h_k.

Remark 5.1. In this computation, α is the angle between the normal in $\mathbb{R}^{k+1}$ to the hyperplane to which we project, and the space coordinate hyperplane; The value of the even $k-$homogeneous cone-symmetric/antisymmetric valuation in $\mathbb{R}^n$ on $C_{n,\epsilon}$ when $\epsilon \neq 0$ is given by $f^{\pm}_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)d\alpha)$ by Proposition 3.7 since the singular support (in fact, the support) of the surface area measure of $C_{n,\epsilon}$ is disjoint from the light cone.

Remark 5.2. We observe for the following that h^+_k, admits a real analytic extension to S^1, and if k is odd then also h^-_k admits a real analytic extension to $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The same holds for $h^{\pm}_{k,\epsilon}$, and in the corresponding cases it holds in the C^∞ topology that

$$\lim_{\epsilon\to 0^\pm} h^{\pm}_{k,\epsilon} = h^{\pm}_k$$

It follows that for any continuous valuation ϕ with generalized Crofton measure $f^{\pm}_{n,k}$, one may write

$$\phi(C^n) = \lim_{\varepsilon\to 0^+}\phi(C_{n,\varepsilon}) = \lim_{\varepsilon\to 0^+} f^{\pm}_{-\frac{n+1}{2}}(h^+_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)) = f^{\pm}_{-\frac{n+1}{2}}(h^+_k(\alpha)g_{n,n-k}(\alpha))$$

and if n is odd then also

$$\phi(C^n) = f^{\pm}_{-\frac{n+1}{2}}(h^-_k(\alpha)g_{n,n-k}(\alpha))$$

5.2 Applying the generalized valuations to the double cone

Proposition 5.3. *(Reduction to $k = n-2$) If for every $n \geq 3$ there exists no continuous even G-invariant $(n-2)$-homogeneous valuation, then there exists no continuous even G-invariant j-homogeneous valuation for $j < n-2$.*

Proof. Let $\phi \in Val^+_j(\mathbb{R}^n)^{SO^+(n-1,1)}$ be such a valuation with $j < n-2$. By our assumption, if Λ is any $(n-2)$-subspace s.t. $Q|_\Lambda$ has mixed signature then $\phi|_\Lambda = 0$. Since every j-dimensional subspace is contained in some Λ as above, we conclude that $Kl_j(\phi) = 0$, and therefore $\phi = 0$. Q.E.D.

Thus we may assume from now on that $k = n-2$, and prove non-extendibility of the corresponding valuations.

Proposition 5.4. *(Odd n, light cone support). For odd n, an $n-2$-homogeneous even valuation ϕ on $\mathbb{R}^n$ having generalized Crofton measure $f \in \Gamma^{-\infty}(E^{n,k})^G$ supported on the light cone, cannot be extended by continuity to all $SO(n-1)$-invariant compact convex bodies.*

Proof. Assume, on the contrary, that this can be accomplished. We will show that ϕ does not extend to the double cone by continuity. Recall from 4.18 that a valuation ϕ as above can occur only for odd n, and by Remark 4.12, it is cone-symmetric if $n \equiv 1 \mod 4$ and cone-antisymmetric otherwise. By Remark 5.1, we may evaluate the valuation on $C_{n,\epsilon}$ by $\phi(C_{n,\epsilon}) = f(h_k(\alpha; C_{n,\epsilon})g_{n,n-k})$. Therefore,

$$\phi(C^n) = \lim_{\epsilon\to 0} f(h_k(\alpha; C_{n,\epsilon})g_{n,n-k})$$

Write

$$f(hg_{n,n-k}) = \sum_{j=0}^{m} c_j h^{(j)}(\frac{\pi}{4})$$

with $m = \frac{n-1}{2}$ (note that the derivatives of $g_{n,n-k}$ are now incorporated into the coefficients c_j). Note that $c_m \neq 0$ since $g_{n,n-k}(\frac{\pi}{4}) \neq 0$. We will show that the limits $\lim_{\epsilon\to 0^+} f(h_{k,\epsilon}(\alpha, C)g_{n,n-k})$ and $\lim_{\epsilon\to 0^-} f(h_{k,\epsilon}(\alpha, C)g_{n,n-k})$ are finite and different from one another, thus arriving at a contradiction. Equivalently, since $\lim_{\epsilon\to 0^+} h_{k,\epsilon} = h_k^+$ in the $C^\infty[-\frac{3\pi}{8}, \frac{3\pi}{8}]$ topology, we will show that

$$\lim_{\epsilon\to 0^-}\Big(f(h^+_{k,\epsilon}(\alpha, C)g_{n,n-k}) - f(h^-_{k,\epsilon}(\alpha, C)g_{n,n-k})\Big) = \lim_{\epsilon\to 0^-} f((h^+_{k,\epsilon}(\alpha, C) - h^-_{k,\epsilon}(\alpha, C))g_{n,n-k})\Big)$$

is non-zero. Denote $v_\epsilon(\alpha) = h^+_{k,\epsilon}(\alpha, C) - h^-_{k,\epsilon}(\alpha, C)$ and $u_\epsilon(\alpha) = (\sin\alpha)^{-1}v_\epsilon(\alpha)$. Consider first the case $n > 3$. Then

$$u_\epsilon(\alpha) = 2\eta\int_{\eta\tan\alpha}^{1}(1-t^2)^{\frac{k-3}{2}}dt - \frac{2}{k-1}\cot\alpha(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}$$

where as before $\eta = \tan(\frac{\pi}{4}+\epsilon)$. It suffices to prove that $\lim_{\epsilon\to 0^-} u_\epsilon^{(j)}(\frac{\pi}{4}) = 0$ for $j \le m-1$, and is non-zero for $j = m$. Indeed, $\lim_{\epsilon\to 0^-} u_\epsilon(\frac{\pi}{4}) = 0$, and

$$u'_\epsilon(\alpha) = \frac{2}{k-1}\frac{(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}}{\sin^2\alpha}$$

Since $k = n-2$, the numerator is a polynomial in $\tan^2\alpha$ with coefficients depending on ϵ, and we conclude that $u'_\epsilon \to \frac{2}{k-1}\frac{1}{\sin^2\alpha}(1-\tan^2\alpha)^{\frac{k-1}{2}}$ in $C^\infty[-\frac{3\pi}{8}, \frac{3\pi}{8}]$. Since

$$(1-\tan^2\alpha)^{\frac{k-1}{2}} = (1-(1+4(\alpha-\frac{\pi}{4})+o(\alpha-\frac{\pi}{4})))^{\frac{k-1}{2}} = (-4)^{\frac{k-1}{2}}(\alpha-\frac{\pi}{4})^{\frac{k-1}{2}}+o((\alpha-\frac{\pi}{4})^{\frac{k-1}{2}})$$

and $\frac{k-1}{2} = \frac{n-3}{2} = m-1$, it follows that

$$\Big((1-\tan^2\alpha)^{m-1}\Big)^{(m-1)}(\frac{\pi}{4}) = (-4)^{m-1}(m-1)!$$

implying the claim.

Now assume $n = 3$ so $k = 1$. Then $v_\epsilon(\alpha) = \eta\cos\alpha - \sin\alpha$, where again $\eta = \tan(\frac{\pi}{4}+\epsilon)$. Since $\lim_{\epsilon\to 0^-} v_\epsilon(\frac{\pi}{4}) = 0$ while $\lim_{\epsilon\to 0^-} v'_\epsilon(\frac{\pi}{4}) = \lim_{\epsilon\to 0^-} -\sin\alpha\eta - \cos\alpha = -\sqrt{2} \neq 0$, the claim follows. Q.E.D.

Remark 5.5. We note for the following that for odd values of n, both $\lim_{\epsilon\to 0^+} f(h_{k,\epsilon}(\alpha,C)g_{n,n-k})$ and $\lim_{\epsilon\to 0^-} f(h_{k,\epsilon}(\alpha,C)g_{n,n-k})$ are finite, where f is the unique G-invariant generalized Crofton measure supported on the light cone.

Proposition 5.6. *(Odd n) For odd n, no $n-2$-homogeneous valuation $\phi \in Val_{n-2}^+(\mathbb{R}^n)^G$ exists.*

Proof. Denote $k = n-2$. Assume first that ϕ is either pure cone-symmetric or cone-antisymmetric, according to $n \mod 4$, such that it is not supported on the light cone.

First, assume $n \equiv 1 \mod 4$, so $n \geq 5$ and $k \geq 3$. Then $\frac{n+1}{2}$ is odd, and $\phi = Cr(f^-_{n,k})$.

$$\phi(C_n) = \lim_{\epsilon\to 0^+} \phi(C_{n,\epsilon}) = \lim_{\epsilon\to 0^+} f^-_{-\frac{n+1}{2}}(h_k(\alpha; C_{n,\epsilon})g_{n,n-k})$$

Note that $h_k(\alpha; C_{n,\epsilon}) = C\eta \sin\alpha$ near $\alpha = \frac{\pi}{4}$, and so all derivatives at $\alpha = \frac{\pi}{4}$ of $h_k(\alpha; C_{n,\epsilon})g_{n,n-k}$ converge to a finite limit as $\epsilon \to 0^+$. Write for an arbitrary function H on S^1,

$$N_-(\alpha; H) = \frac{H(\alpha) - H(\frac{\pi}{2}-\alpha) - 2(H'(\frac{\pi}{4})(\alpha-\frac{\pi}{4}) + \frac{1}{3!}H^{(3)}(\frac{\pi}{4})(\alpha-\frac{\pi}{4})^3 + ... + \frac{1}{(2m+1)!}H^{(2m+1)}(\frac{\pi}{4})(\alpha-\frac{\pi}{4})^{2m+1})}{|\cos 2\alpha|^{\frac{n+1}{2}}}$$

where $m = \frac{n-1}{4}$. Denote $H_\epsilon(\alpha) = h_k(\alpha; C_{n,\epsilon})g_{n,n-k}(\alpha)$. We will show that the integral

$$I_-(H_\epsilon) = \int_0^{\frac{\pi}{4}} N_-(\alpha; H_\epsilon)d\alpha$$

which equals $\phi(C_{n,\epsilon})$ up to bounded summands, diverges as $\epsilon \to 0^+$. Then

$$I_-(H_\epsilon) = \int_0^{\frac{\pi}{4}-\epsilon} \frac{h^-_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha) - h^+_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}}d\alpha + \int_0^{\frac{\pi}{4}} N_-(\alpha; h^+_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha))d\alpha$$

Now the second integral is bounded (uniformly in ϵ), for instance by $C|\int_0^{\frac{\pi}{4}} N_-(\alpha; h^+_k(\alpha)g_{n,n-k}(\alpha))d\alpha|$.

We will show that the first summand is unbounded. Calculate first that

$$\frac{d}{d\alpha}\Big(\frac{h^-_{k,\epsilon}(\alpha) - h^+_{k,\epsilon}(\alpha)}{\sin\alpha}\Big) = 2\frac{d}{d\alpha}\left(\frac{2}{k-1}\cot\alpha(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}} - \left(\int_{\eta\tan\alpha}^1 (1-t^2)^{\frac{k-3}{2}}dt\right)\eta\right) =$$

$$= \frac{\eta^2}{\cos^2\alpha}(1-\eta^2\tan^2\alpha)^{\frac{k-3}{2}} - (1-\eta^2\tan^2\alpha)^{\frac{k-3}{2}}\Big(\frac{2}{k-1}\frac{1-\eta^2\tan^2\alpha}{\sin^2\alpha} + \frac{\eta^2}{\cos^2\alpha}\Big) =$$

$$= -\frac{2}{k-1}\frac{(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}}}{\sin^2\alpha} \tag{9}$$

which is negative. Since $h^+_{k,\epsilon}(\frac{\pi}{4}-\epsilon) = h^-_{k,\epsilon}(\frac{\pi}{4}-\epsilon)$, it follows that $h^-_{k,\epsilon}(\alpha) - h^+_{k,\epsilon}(\alpha) > 0$ in $(0, \frac{\pi}{4}-\epsilon)$. Now

$$\int_0^{\frac{\pi}{4}-\epsilon} \frac{h^-_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha) - h^+_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}}d\alpha \geq C_n + c_n\int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon} \frac{h^-_{k,\epsilon}(\alpha) - h^+_{k,\epsilon}(\alpha)}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}}d\alpha$$

$$\int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon} \frac{h^-_{k,\epsilon}(\alpha) - h^+_{k,\epsilon}(\alpha)}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}}d\alpha =$$

$$\geq c'_n \int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon} \frac{1}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}} \frac{h^-_{k,\epsilon}(\alpha) - h^+_{k,\epsilon}(\alpha)}{\sin\alpha} d\alpha$$

Now integrate by parts: we integrate $(\frac{\pi}{4}-\alpha)^{-\frac{n+1}{2}}$ and differentiate the other term. The boundary term is bounded uniformly in ϵ, and we already computed the derivative of $\frac{h^-_{k,\epsilon}(\alpha)-h^+_{k,\epsilon}(\alpha)}{\sin\alpha}$ in equation 9. The resulting integral thus equals

$$= c_{n,k} \int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon} \frac{(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}} d\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n-1}{2}} \sin^2\alpha} \geq c'_{n,k} \int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon} \frac{(1-\eta^2\tan^2\alpha)^{\frac{k-1}{2}} d\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n-1}{2}}}$$

Now $1-\eta^2\tan^2\alpha \geq \frac{1}{4}(\alpha_\epsilon - \alpha)$ so the integral is bounded from below by

$$c'_{n,k} \int_{\frac{\pi}{5}}^{\frac{\pi}{4}-\epsilon} \frac{(\alpha_\epsilon-\alpha)^{\frac{k-1}{2}} d\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n-1}{2}}} = c'_{n,k} \int_0^{\frac{\pi}{4}-\epsilon-\frac{\pi}{5}} \frac{t^{\frac{k-1}{2}} dt}{(\epsilon+t)^{\frac{n-1}{2}}} \geq c'_{n,k} \int_0^{\frac{\pi}{100}} \frac{t^{\frac{k-1}{2}} dt}{(\epsilon+t)^{\frac{n-1}{2}}}$$

Finally, the limit

$$\lim_{\epsilon\to 0^+} \int_0^{\frac{\pi}{100}} \frac{t^{\frac{k-1}{2}} dt}{(\epsilon+t)^{\frac{n-1}{2}}} = \infty$$

is infinite. Thus $I_-(H_\epsilon)$ is unbounded as $\epsilon \to 0^+$, i.e $\phi(C_{n,\epsilon}) \to \infty$.

Now assume $n \equiv 3 \mod 4$ and $n \geq 7$, so $k \geq 5$ and ϕ corresponds to $f^+_{n,k}$. For an arbitrary function H on S^1 define $N_+(\alpha; H)$ by

$$N_+(\alpha;H) = \frac{H(\alpha)+H(\frac{\pi}{2}-\alpha) - 2(H(\frac{\pi}{4}) + \frac{1}{2!}H^{(2)}(\frac{\pi}{4})(\alpha-\frac{\pi}{4})^2 + \ldots + \frac{1}{(2m)!}H^{(2m)}(\frac{\pi}{4})(\alpha-\frac{\pi}{4})^{2m})}{|\cos 2\alpha|^{\frac{n+1}{2}}}$$

where $m = \frac{n-3}{4}$. Exactly as before, the integral

$$I_+(H_\epsilon) = \int_0^{\frac{\pi}{4}} N_+(\alpha; H_\epsilon) d\alpha$$

is unbounded as $\epsilon \to 0^+$, i.e $\phi(C_{n,\epsilon}) \to \infty$.

Let us compute separately the case of $k=1$ and $n=3$.
Then

$$I_-(H_\epsilon) = \int_0^{\frac{\pi}{4}-\epsilon} \frac{\cos\alpha g_{n,n-1}(\alpha) - \eta\sin\alpha g_{n,n-1}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}} d\alpha + \int_0^{\frac{\pi}{4}} N_-(\alpha; \eta\sin\alpha g_{n,n-1}(\alpha)) d\alpha$$

where

$$N_-(\alpha;H) = \frac{H(\alpha) - H(\frac{\pi}{2}-\alpha) - 2H'(\frac{\pi}{4})(\alpha-\frac{\pi}{4})}{|\cos 2\alpha|^{\frac{n+1}{2}}}$$

Now the second integral is bounded (uniformly in ϵ), for instance by $2|\int_0^{\frac{\pi}{4}} N_-(\alpha; \sin\alpha g_{n,n-1}(\alpha))d\alpha|$. The first integrand is non-negative, and since $g_{n,n-1}(\alpha) \geq c_n$ for $\alpha \in [\frac{\pi}{10}, \frac{\pi}{4}]$ while $\cos 2\alpha \leq c|\alpha - \frac{\pi}{4}|$ in that interval, we get

$$\int_0^{\frac{\pi}{4}-\epsilon} \frac{\cos\alpha g_{n,n-1}(\alpha) - \eta\sin\alpha g_{n,n-1}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}} d\alpha \geq c \int_{\frac{\pi}{10}}^{\frac{\pi}{4}-\epsilon} \frac{\cos\alpha - \eta\sin\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}} d\alpha \geq$$

$$\geq c \int_0^{\frac{\pi}{4}-\epsilon} \frac{\cos\alpha - \eta\sin\alpha}{(\frac{\pi}{4}-\alpha)^{\frac{n+1}{2}}} d\alpha$$

The function $\cos\alpha - \eta\sin\alpha$ is decreasing and concave for $0 \leq \alpha \leq \frac{\pi}{4} - \epsilon$, so $\cos\alpha - \eta\sin\alpha \geq 1 - \frac{\alpha}{\frac{\pi}{4}-\epsilon}$ for $0 \leq \alpha \leq \frac{\pi}{4} - \epsilon$. Therefore

$$\int_0^{\frac{\pi}{4}-\epsilon} \frac{\cos\alpha - \eta\sin\alpha}{(\frac{\pi}{4} - \alpha)^{\frac{n+1}{2}}} d\alpha \geq \frac{1}{\frac{\pi}{4} - \epsilon} \int_0^{\frac{\pi}{4}-\epsilon} (\frac{\pi}{4} - \alpha)^{-\frac{n-1}{2}} d\alpha + (1 - \frac{\pi/4}{\pi/4 - \epsilon}) \int_0^{\frac{\pi}{4}-\epsilon} (\frac{\pi}{4} - \alpha)^{-\frac{n+1}{2}} d\alpha$$

recalling that $n = 3$, that equals

$$-\frac{1}{\frac{\pi}{4} - \epsilon} \log\frac{\epsilon}{\frac{\pi}{4}} + \Big(1 - \frac{\pi/4}{\frac{\pi}{4} - \epsilon}\Big)\frac{2}{3-1}\epsilon^{-\frac{3-1}{2}} = -\frac{1}{\frac{\pi}{4} - \epsilon}\log\epsilon + O(1)$$

Thus for all $k \geq 1$, $I_-(H_\epsilon)$ is unbounded as $\epsilon \to 0^+$.

Finally, consider a general $f = af^+_{n,k} + bf^-_{n,k}$, given by a linear combination of pure cone-symmetric and cone-antisymmetric sectios, and assume it corresponds to a continuous valuation. Then by the preceding argument and Proposition 5.4, we must have both $a \neq 0$ and $b \neq 0$. When evaluated on H_ϵ, this would diverge as $\epsilon \to 0^+$, since the light cone-supported summand has a limit by Remark 5.5, while the other summand diverges as was just proved. Q.E.D.

Proposition 5.7. *(Even n, reduction to time-supported valuation) For even n, an $(n-2)$-homogeneous valuation $\phi \in Val^+_{n-2}(\mathbb{R}^n)^G$ on $\mathbb{R}^n$, if exists, has generalized Crofton measure equal to a multiple of $f^T_{n,n-2}$.*

Proof. Denote $k = n-2$, assume ϕ corresponds to $f = af^T_{n,k} + bf^S_{n,k}$.

$$\phi(C_n) = \lim_{\epsilon\to 0^+} \phi(C_{n,\epsilon}) = \lim_{\epsilon\to 0^+} f^T_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k})$$

Note that $h_{k,\epsilon}(\alpha) = C\eta\sin\alpha$ for $|\alpha| > \frac{\pi}{4} - \epsilon$, and so all derivatives at $\alpha = \frac{\pi}{4}$ of $h_{k,\epsilon}(\alpha)g_{n,n-k}$ converge to a finite limit as $\epsilon \to 0^+$and likewise $\lim_{\epsilon\to 0^+} f^T_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k})$ is finite. We will show that $\lim_{\epsilon\to 0^+} f^S_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k})$ is inifinite, implying $b = 0$.

Denote $H_\epsilon(\alpha) = h_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)$. Write for an arbitrary function H on S^1,

$$N(\alpha; H) = \frac{H(\alpha) - (H(\frac{\pi}{4}) + \frac{1}{1!}H^{(1)}(\frac{\pi}{4})(\alpha - \frac{\pi}{4}) + ... + \frac{1}{m!}H^{(m)}(\frac{\pi}{4})(\alpha - \frac{\pi}{4})^m)}{|\cos 2\alpha|^{\frac{n+1}{2}}}$$

where $m = \frac{n-2}{2}$. The integral

$$I(H_\epsilon) = \int_0^{\frac{\pi}{4}} N(\alpha; H_\epsilon)d\alpha$$

equals $f^S_{-\frac{n+1}{2}}(h_{k,\epsilon}(\alpha)g_{n,n-k})$ up to summands corresponding to derivatives of $h_{k,\epsilon}(\alpha)g_{n,n-k}$ at the light cone, of order up to m. Those derivatives are uniformly bounded as $\epsilon \to 0^+$, since $h_{k,\epsilon}(\alpha) \to h_k(\alpha)$ in the $C^m(S^1)$ topology by the remark following Proposition 3.6).

We will show that $I(H_\epsilon)$ diverges as $\epsilon \to 0^+$. Write

$$I(H_\epsilon) = \int_0^{\frac{\pi}{4}-\epsilon} \frac{h^-_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha) - h^+_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha)}{|\cos 2\alpha|^{\frac{n+1}{2}}} d\alpha + \int_0^{\frac{\pi}{4}} N(\alpha; h^+_{k,\epsilon}(\alpha)g_{n,n-k}(\alpha))d\alpha$$

Now the second integral is bounded (uniformly in ϵ), for instance by $C|\int_0^{\frac{\pi}{4}} N(\alpha; h^+_k(\alpha)g_{n,n-k}(\alpha))d\alpha|$, and the first summand is unbounded, exactly as in the case of odd n before . This concludes the proof. Q.E.D.

Proposition 5.8. *(Non-existence of time-supported valuation with $k = n-2$) For n even, $Cr(f^T_{n,n-2})$ is not a continuous valuation.*

Proof. Denote $k = n-2$, $m = \frac{k}{2} = \frac{n}{2} - 1$, and assume $\phi = Cr(f^T_{n,n-2})$ is a continuos valuation. As before $H_\epsilon(\alpha) = h_{k,\epsilon} g_{n,n-k}(\alpha)$. By Remark 3.6, $h_{k,\epsilon}(\alpha) \to h_k(\alpha)$ as $\epsilon \to 0$ in $C^m(S^1)$.
Introduce the notations

$$J_j(\alpha; H, \alpha_0) = H(\alpha_0) + \frac{1}{1!} H^{(1)}(\alpha_0)(\alpha - \alpha_0) + ... + \frac{1}{j!} H^{(j)}(\alpha_0)(\alpha - \alpha_0)^j$$

$$N(\alpha; H, j) = \frac{H(\alpha) - J_j(\alpha; H, \frac{\pi}{4})}{|\cos 2\alpha|^{j+\frac{3}{2}}}$$

and

$$I(u) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} N(\alpha; u, m) d\alpha$$

Observe that $H_\epsilon \to H$ in $C^m(S^1)$ as well, so all the derivatives satisfy $H_\epsilon^{(j)}(\frac{\pi}{4}) \to H^{(j)}(\frac{\pi}{4})$ for $j \le m$ as $\epsilon \to 0$. We will show that

$$\lim_{\epsilon \to 0^+} f^T_{-\frac{n+1}{2}}(H_\epsilon) \neq \lim_{\epsilon \to 0^-} f^T_{-\frac{n+1}{2}}(H_\epsilon)$$

Equivalently, due to C^m convergence, we will show that $I(H_\epsilon)$ has different one-sided limits.
Denote

$$u_\epsilon(\alpha) = \frac{h_{k,\epsilon}(\alpha)}{\sin \alpha}$$

Recall that

$$u_\epsilon(\alpha) = \begin{cases} A_k \eta, \alpha \ge \frac{\pi}{4} - \epsilon \\ A_k \eta - 2\eta \int_{\eta \tan \alpha}^1 (1-t^2)^{\frac{k-3}{2}} dt + \frac{2}{k-1} \cot \alpha (1 - \eta^2 \tan^2 \alpha)^{\frac{k-1}{2}}, 0 \le \alpha \le \frac{\pi}{4} - \epsilon \end{cases}$$

where $A_k = \int_{-\pi/2}^{\pi/2} \cos^{k-2} \phi d\phi$, implying

$$\lim_{\epsilon \to 0^+} I(u_\epsilon) = \lim_{\epsilon \to 0^+} I(A_k \eta) = 0$$

Now write $H_\epsilon = t(\alpha) u_\epsilon(\alpha)$ where $t(\alpha) = g_{n,n-k}(\alpha) \sin \alpha$. According to Lemma A.1, we may write

$$H(\alpha) - J_j(\alpha; H, \frac{\pi}{4}) = t(\frac{\pi}{4})(u_\epsilon(x) - J_m(\alpha; u_\epsilon, \frac{\pi}{4})) + u_\epsilon(\alpha) R_{m+1}(\alpha) + O(C_\epsilon |\alpha - \frac{\pi}{4}|^{m+1})$$

where $R_{m+1}(\alpha) = t(\alpha) - J_m(\alpha; t, \frac{\pi}{4})$, and the constant C_ϵ in the error term is bounded by

$$C_m \sup_{1 \le j \le m} |u_\epsilon^{(j)}|$$

with

$$C_m = m \sup_{0 \le j \le m+1} |(g_{n,n-k}(\alpha) \sin \alpha)^{(j)}|$$

where everywhere $\alpha \in [0, \frac{\pi}{2}]$. By the convergence of $u_\epsilon(\alpha) \to A_k$ in $C^m[\frac{\pi}{4}, \frac{\pi}{2}]$, we conclude that $C_\epsilon \to 0$ as $\epsilon \to 0$.
Since $|R_{m+1}(\alpha)| \le C|\alpha - \frac{\pi}{4}|^{m+1}$, and u_ϵ converges in $C[\frac{\pi}{4}, \frac{\pi}{2}]$, the integral

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{u_\epsilon(\alpha) R_{m+1}(\alpha)}{|\cos 2\alpha|^{m+\frac{3}{2}}} d\alpha$$

has a limit as $\epsilon \to 0$. Also, the integral

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{O(C_\epsilon|\alpha-\frac{\pi}{4}|^{m+1})}{|\cos 2\alpha|^{m+\frac{3}{2}}} d\alpha$$

converges to 0 as $\epsilon \to 0$. We conclude that $I(H_\epsilon) - t(\frac{\pi}{4})I(u_\epsilon)$ converges, and thus it suffices to show that the functional $I(u_\epsilon)$ has different one-sided limits. We will verify that

$$\lim_{\epsilon\to 0^-} I(u_\epsilon) \neq 0$$

From now on $\epsilon < 0$. We will use the approximations

$$\eta = \tan(\frac{\pi}{4}+\epsilon) = 1 + 2\epsilon + O(\epsilon^2)$$

$$1-\eta^4 = -8\epsilon + O(\epsilon^2)$$

$$(1-\eta^2)^{\frac{1}{2}} = (-4\epsilon + O(\epsilon^2))^{\frac{1}{2}} = 2|\epsilon|^{\frac{1}{2}} + O(|\epsilon|)$$

Then for $\alpha < \frac{\pi}{4} - \epsilon$,

$$u_\epsilon'(\alpha) = -\frac{2}{k-1}(1-\eta^2\tan^2\alpha)^{m-\frac{1}{2}}\frac{1}{\sin^2\alpha}$$

It follows by induction that for $\alpha \in (\frac{\pi}{4}+\epsilon, \frac{\pi}{4}-\epsilon)$ and $j \geq 1$,

$$u_\epsilon^{(j)}(\alpha) = (-1)^j \frac{2}{k-1}\frac{1}{\sin^2\alpha}\frac{1}{2^{j-1}}\frac{(2m-1)!!}{(2m-2j+1)!!}\eta^{2j-2}(1-\eta^2\tan^2\alpha)^{m+\frac{1}{2}-j}\left(\frac{2\tan\alpha}{\cos^2\alpha}\right)^{j-1}+$$

$$+O\Big((1-\eta^2\tan^2\alpha)^{m+\frac{3}{2}-j}\Big) =$$

$$= (-1)^j\frac{2}{k-1}\frac{(2m-1)!!}{(2m-2j+1)!!}\eta^{2j-2}\frac{\sin^{j-3}\alpha}{\cos^{3j-3}\alpha}(1-\eta^2\tan^2\alpha)^{m+\frac{1}{2}-j}+O\Big((1-\eta^2\tan^2\alpha)^{m+\frac{3}{2}-j}\Big)$$

in particular, for $1 \leq j \leq m$ and $\alpha \in (\frac{\pi}{4}+\epsilon, \frac{\pi}{4}-\epsilon)$, $|1-\eta^2\tan^2\alpha| = O(|\epsilon|)$ so

$$|u_\epsilon^{(j)}(\alpha)| = O(|\varepsilon|^{m+\frac{1}{2}-j}) \tag{10}$$

It therefore also holds that

$$|u_\epsilon(\frac{\pi}{4}) - A_k\eta| = |u_\epsilon(\frac{\pi}{4}) - u_\varepsilon(\frac{\pi}{4}-\epsilon)| = O(|\varepsilon|^{m+\frac{1}{2}}) \tag{11}$$

Write

$$I(u_\epsilon) = \int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{A_k\eta - J_m(\alpha; u_\epsilon, \frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}}w(\alpha)d\alpha + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon}\frac{u_\epsilon(\alpha) - J_m(\alpha;u_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}}w(\alpha)d\alpha \tag{12}$$

where

$$w(\alpha) = \frac{|\alpha-\frac{\pi}{4}|^{m+\frac{3}{2}}}{|\cos 2\alpha|^{m+\frac{3}{2}}}$$

is a C^∞, strictly positive function in $[0, \frac{\pi}{2}]$. Now integrate by parts: we integrate the denominator and differentiate the numerator.

$$\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{A_k\eta - J_m(\alpha;u_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}}w(\alpha)d\alpha = -\frac{1}{m+\frac{1}{2}}\frac{A_k\eta - J_m(\frac{\pi}{2};u_\epsilon,\frac{\pi}{4})}{(\frac{\pi}{4})^{m+\frac{1}{2}}}w(\frac{\pi}{2})$$

$$+\frac{1}{m+\frac{1}{2}}\frac{A_k\eta - J_m(\frac{\pi}{4}-\epsilon;u_\epsilon,\frac{\pi}{4})}{|\epsilon|^{m+\frac{1}{2}}}w(\frac{\pi}{4}-\epsilon)+\frac{1}{m+\frac{1}{2}}\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{J_{m-1}(\alpha;-u_\epsilon',\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w(\alpha)d\alpha+$$

$$+\frac{1}{m+\frac{1}{2}}\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{A_k\eta - J_m(\alpha;u_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w'(\alpha)d\alpha$$

the first summand is $o(1)$ as $\epsilon \to 0^-$, since $u_\epsilon \to A_k \in C^m[\frac{\pi}{4},\frac{\pi}{2}]$, so $J_m(\frac{\pi}{2};u_\epsilon,\frac{\pi}{4}) \to J_m(\frac{\pi}{2};A_k,\frac{\pi}{4}) = A_k$ as $\epsilon \to 0^-$. Let us verify that the last summand is also $o(1)$. Indeed

$$\left|\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{A_k\eta - J_m(\alpha;u_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w'(\alpha)d\alpha\right| \le C\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\left(\frac{|A_k\eta - u_\epsilon(\frac{\pi}{4})|}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}+\frac{1}{j!}\sum_{j=1}^{m}\frac{|u_\epsilon^{(j)}(\frac{\pi}{4})|}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}-j}}\right)d\alpha$$

This can be integrated explicitly. The terms corresponding to $\frac{\pi}{2}$ are all $o(1)$ again since $u_\epsilon \to A_k \in C^m[\frac{\pi}{4},\frac{\pi}{2}]$, while the terms corresponding to $\frac{\pi}{4}-\epsilon$ are all $O(|\epsilon|)$ by estimates 10 and 11. Therefore,

$$\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{A_k\eta - J_m(\alpha;u_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}}w(\alpha)d\alpha =$$

$$=\frac{1}{m+\frac{1}{2}}\left(\frac{A_k\eta - J_m(\frac{\pi}{4}-\epsilon;u_\epsilon,\frac{\pi}{4})}{|\epsilon|^{m+\frac{1}{2}}}w(\frac{\pi}{4}-\epsilon)+\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{J_{m-1}(\alpha;-u'_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w(\alpha)d\alpha\right)+o(1)$$

Similarly, we may integrate by parts the second summand of $I(u_\epsilon)$ in equation 12 as follows:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon}\frac{u_\epsilon(\alpha) - J_m(\alpha;u_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{3}{2}}}w(\alpha)d\alpha = -\frac{1}{m+\frac{1}{2}}\frac{A_k\eta - J_m(\frac{\pi}{4}-\epsilon;u_\epsilon,\frac{\pi}{4})}{|\epsilon|^{m+\frac{1}{2}}}w(\frac{\pi}{4}-\epsilon)+$$

$$+\frac{1}{m+\frac{1}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon}\frac{u'_\epsilon(\alpha) + J_{m-1}(\alpha;-u'_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w(\alpha)d\alpha$$

the $\frac{\pi}{4}$-boundary term vanishes since u_ϵ is C^∞ near $\frac{\pi}{4}$. Thus

$$I(u_\epsilon) = \frac{1}{m+\frac{1}{2}}\left(\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{J_{m-1}(\alpha;-u'_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w(\alpha)d\alpha+\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon}\frac{u'_\epsilon(\alpha) - J_{m-1}(\alpha;u'_\epsilon,\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{m+\frac{1}{2}}}w(\alpha)d\alpha\right)+o(1)$$

so we should show that the expression in the brackets does not vanish as $\epsilon \to 0^-$. Repeatedly applying integration by parts as we did for equation 12, we end up having to show that

$$J(\epsilon) = \int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}}\frac{-u_\epsilon^{(m)}(\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}}w(\alpha)d\alpha + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon}\frac{u_\epsilon^{(m)}(\alpha) - u_\epsilon^{(m)}(\frac{\pi}{4})}{(\alpha-\frac{\pi}{4})^{\frac{3}{2}}}w(\alpha)d\alpha$$

does not converge to 0 as $\epsilon \to 0^-$.

Recall that

$$u_\epsilon^{(m)}(\alpha) = (-1)^m\frac{2}{k-1}(2m-1)!!\eta^{2m-2}(1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha}+O\Big((1-\eta^2\tan^2\alpha)^{3/2}\Big)$$

in particular,

$$u_\epsilon^{(m)}(\frac{\pi}{4}) = (-1)^m\frac{2}{k-1}(2m-1)!!(1-\eta^2)^{\frac{1}{2}}\eta^{2m-2}2^m + O(|\epsilon|^{3/2}) =$$

$$= (-1)^m\frac{2}{k-1}(2m-1)!!\eta^{2m-2}2^{m+1}|\epsilon|^{1/2} + O(|\epsilon|^{3/2})$$

We will also need the finer estimate

$$u_\epsilon^{(m)}(\alpha) - u_\epsilon^{(m)}(\frac{\pi}{4}) = (-1)^m \frac{2}{k-1}(2m-1)!!\eta^{2m-2}\Big((1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} - (1-\eta^2)^{\frac{1}{2}}2^m\Big) +$$

$$+O\Big(\alpha - \frac{\pi}{4}\Big)$$

which is obtained by writing

$$u_\epsilon^{(m)}(\alpha) = (-1)^m \frac{2}{k-1}(2m-1)!!\eta^{2m-2}(1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} + s_\epsilon(\alpha)(1-\eta^2\tan^2\alpha)^{3/2}$$

where $s_\epsilon(\alpha) \in C^1(\frac{\pi}{5}, \frac{\pi}{3})$ is uniformly bounded in $C^1(\frac{\pi}{5}, \frac{\pi}{3})$. Then the error term in $u_\epsilon^{(m)}(\alpha) - u_\epsilon^{(m)}(\frac{\pi}{4})$ is easily seen to equal

$$+O\Big((1-\eta^2)^{3/2} - (1-\eta^2\tan^2\alpha)^{3/2}\Big) + O\Big(\alpha - \frac{\pi}{4}\Big)$$

and since $(1-\eta^2\tan^2\alpha)^{3/2}$ is $C^1(\frac{\pi}{5}, \frac{\pi}{3})$ and uniformly bounded, one has

$$(1-\eta^2)^{3/2} - (1-\eta^2\tan^2\alpha)^{3/2} = O\Big(\alpha - \frac{\pi}{4}\Big)$$

Integrating the first summand of $J(\epsilon)$ by parts, we get that

$$\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{2}} \frac{-u_\epsilon^{(m)}(\frac{\pi}{4})}{(\alpha - \frac{\pi}{4})^{\frac{3}{2}}} w(\alpha)d\alpha = -u_\epsilon^{(m)}(\frac{\pi}{4})\frac{2}{|\epsilon|^{\frac{1}{2}}}w(\frac{\pi}{4}) + o(1)$$

$$= (-1)^{m+1}\frac{2}{k-1}(2m-1)!!\eta^{2m-2}w(\frac{\pi}{4})\Big(2^{m+2} + o(1)\Big)$$

while

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{u_\epsilon^{(m)}(\alpha) - u_\epsilon^{(m)}(\frac{\pi}{4})}{(\alpha - \frac{\pi}{4})^{\frac{3}{2}}} w(\alpha)d\alpha =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{(-1)^m\frac{2}{k-1}(2m-1)!!\eta^{2m-2}\Big((1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} - (1-\eta^2)^{\frac{1}{2}}2^m\Big)}{(\alpha - \frac{\pi}{4})^{\frac{3}{2}}} w(\alpha)d\alpha + o(1) =$$

$$= (-1)^m\frac{2}{k-1}(2m-1)!!\eta^{2m-2}w(\frac{\pi}{4})\int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{(1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} - (1-\eta^2)^{\frac{1}{2}}2^m}{(\alpha - \frac{\pi}{4})^{\frac{3}{2}}} d\alpha + o(1)$$

So it remains to show that

$$-2^{m+2} + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{(1-\eta^2\tan^2\alpha)^{\frac{1}{2}}\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} - (1-\eta^2)^{\frac{1}{2}}2^m}{(\alpha - \frac{\pi}{4})^{\frac{3}{2}}} d\alpha \nrightarrow 0$$

Since

$$\frac{\sin^{m-3}\alpha}{\cos^{3m-3}\alpha} = 2^m + O(\alpha - \frac{\pi}{4})$$

this boils down to

$$-4 + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}-\epsilon} \frac{(1-\eta^2\tan^2\alpha)^{\frac{1}{2}} - (1-\eta^2)^{\frac{1}{2}}}{(\alpha - \frac{\pi}{4})^{\frac{3}{2}}} d\alpha \nrightarrow 0$$

The integral is non-positive. This concludes the proof. Q.E.D.

6 Applications

Recently in [21], some negative results on continuity properties of classical constructions in the theory of valuations were proved. We will now explain how some of those results can be seen immediately from the classification of Lorentz-invariant valuations.

6.1 The image of the Klain imbedding is not closed

Denote by $\phi^{\pm}_{n,k} \in Val_k^{ev,-\infty}(V)^G$ the two independent generalized valuations that we found. The generalized Klain sections $Kl(\phi^{\pm}_{n,k}) \in \Gamma(K^{n,k})$ for $1 \leq k \leq n-2$ are in fact continuous sections of the Klain bundle, that do not correspond to a continuous valuation. They do belong to the closure (in the C^0 topology) of the image of the Klain imbedding on continuous valuations.

6.2 The Fourier transform does not extend to continuous valuations

The Fourier transform on smooth even valuations extends to the space of generalized smooth valuations by self-adjointness (see [8]): For $\phi \in Val_k^{ev,-\infty}(V)$, we define $\mathbb{F}\phi \in Val_{n-k}^{ev,-\infty}(V^*) \otimes D(V)$ by letting for all $\psi \in Val_k^{ev,\infty}(V^*)$

$$\langle \mathbb{F}\phi, \psi \rangle = \langle \phi, \mathbb{F}\psi \rangle$$

It is a $GL(V)$-equivariant involution (in the sense that $(\mathbb{F}_{V^*} \otimes Id) \circ \mathbb{F}_V = Id$). Restricting to $G = SO^+(n-1,1)$, we get a G-equivariant involution

$$\mathbb{F} : Val_k^{ev,-\infty}(V) \to Val_{n-k}^{ev,-\infty}(V)$$

which restricts to the usual (G-equivariant) Fourier transform on smooth even valuations.
Let $\phi^{\pm}_{n,n-1} \in Val_{n-1}^{ev}(V)^G$ be the cone-symmetric and cone-antisymmetric continuous valuations that we found. It follows by equivariance that

$$\mathbb{F}(\phi^{\pm}_{n,n-1}) \in Val_1^{ev,-\infty}(V)^G$$

Since $Val_1^{ev,-\infty}(V)^G$ contains no non-trivial continuous valuations when $n \geq 3$, it follows that the Fourier transform does not extend by continuity to continuous valuations for $n \geq 3$.

A A technical lemma

We denote by $J^m(x; f, a)$ the Taylor polynomial of order m for the function f around a.

Lemma A.1. *For $w \in C^\infty(\mathbb{R})$ and $h \in C^m(\mathbb{R})$ it holds in any fixed compact interval I around 0 that*

$$w(x)h(x) - J_m(x; wh, 0) = w(0)(h(x) - J_m(x; h, 0)) + h(x)R_{m+1}(x) + O(|x|^{m+1})$$

as $x \to 0$, where $R_{m+1}(x) = w(x) - J_m(x; w, 0)$. More precisely, if $|h^{(j)}(x)| \leq H_j$ for all $x \in I$ and $0 \leq j \leq m$ and $|w^{(j)}(x)| \leq W$ for all $x \in I$ and $j \leq m+1$ then $O(|x|^{m+1}) \leq C_{m,I}(H_m + ... + H_1)W|x|^{m+1}$.

Proof. Write $J_m(f)$ for $J_m(x; f, 0)$. Then

$$h = J_m(h) + e_1(x)$$

$$w - w(0) = J_m(w - w(0)) + e_2(x)$$

where

$$|e_1(x)| \le c_{m,I} H_m |x|^m$$

$$|e_2(x)| \le c'_{m,I} W |x|^{m+1}$$

so

$$wh = (w - w(0))h + w(0)h = J_m(w - w(0))h + w(0)h + hR_{m+1} =$$

$$= J_m(w - w(0))(J_m(h) + O(H_m|x|^m)) + w(0)h + hR_{m+1} =$$

$$= J_m(w - w(0))J_m(h) + w(0)h + O\Big(H_m W |x|^{m+1}\Big) + hR_{m+1}$$

the last equality since $J_m(w - w(0)) = O(W_1|x|)$. Note that

$$J_m(w - w(0))J_m(h) = J_m((w - w(0))h) + O\Big((H_m + ... + H_1)W|x|^{m+1}\Big) =$$

$$= J_m(wh) - w(0)J_m(h) + O\Big((H_m + ... + H_1)W|x|^{m+1}\Big)$$

so

$$wh = J_m(wh) - w(0)J_m(h) + w(0)h + O((H_m + ... + H_1)W|x|^{m+1}) + hR_{m+1}$$

as claimed. Q.E.D.

References

[1] Alesker, Semyon; *On P. McMullen's conjecture on translation invariant valuations,* Adv. Math. 155 (2000), no. 2, 239-263.

[2] Alesker, Semyon; *The multiplicative structure on polynomial continuous valuations,* Geom. Funct. Anal. 14(1) (2004), 1-26.

[3] Alesker, Semyon; *Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations*, Journal of Differential Geometry 63 (2003), 63-95.

[4] Alesker, Semyon; Bernstein, Joseph; *Range characterization of the cosine transform on higher Grassmannians*, Advances in Math., 184 (2004), 367-379.

[5] Bernig, Andreas; *A Hadwiger-type theorem for the special unitary group,* Geom. Funct. Anal. 19 (2009), no. 2, 356-372.

[6] Bernig, Andreas; *Integral geometry under G_2 and $Spin(7)$*, Israel J. Math. 184 (2011), 301-316.

[7] Bernig, Andreas; *Invariant valuations on quaternionic vector spaces,* Journal de l'Institut de Mathématiques de Jussieu 11 (2012), 467-499.

[8] Bernig, Andreas; Fu, Joseph H. G. *Convolution of convex valuations,* Geom. Dedicata (2006) 123:153-169.

[9] Bernig, Andreas; Fu, Joseph H. G. *Hermitian integral geometry,* Ann. of Math. (2) 173 (2011), no. 2, 907-945.

[10] Fu, Joseph H. G.; *Structure of the unitary valuation algebra*, J. Differential Geom. 72 (2006), no. 3, 509-533.

[11] D.B. Fuks, Cohomology of Infinite-Dimensional Lie Algebras, Contemporary Soviet Mathematics, Steklov Institute, Moscow, 1986.

[12] Goodey, Paul R.; Weil, Wolfgang; *Centrally symmetric convex bodies and the spherical Radon transform*, J. Differential Geom. 35 (1992) 675-688.

[13] Guillemin, Victor; *On some results of Gelfand in integral geometry,* Proc.Symp.Pure Math. 43(1985),149-155.

[14] H. Hadwiger, Vorlesungen ?ber Inhalt, Oberfläche und Isoperimetrie. (German) Springer-Verlag, Berlin-Göttingen-Heidelberg 1957.

[15] Klain, Daniel A.; *A short proof of Hadwiger's characterization theorem,* Mathematika 42 (1995), no. 2, 329-339.

[16] Ludwig, Monika; *Projection bodies and valuations*, Adv. Math. 172 (2002), no. 2, 158-168.

[17] Ludwig, Monika; *Intersection bodies and valuations,* Amer. J. Math. 128 (2006), no. 6, 1409-1428.

[18] Ludwig, Monika; Reitzner, Matthias; *A characterization of affine surface area*, Adv. Math. 147 (1999), no. 1, 138-172.

[19] Ludwig, Monika; Reitzner, Matthias; *A classification of SL(n) invariant valuations,* Ann. of Math. (2) 172 (2010), no. 2, 1219-1267.

[20] McMullen, Peter; *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. (3) 35:1 (1977), 113-135.

[21] Parapatits, Lukas; Wannerer, Thomas; *On the inverse Klain map,* arXiv:1206.5370 [math.MG].

[22] Schneider, Rolf; Convex bodies: the Brunn-Minkowski theory. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1993.

[23] Schneider, Rolf; *Simple valuations on convex bodies*, Mathematika 43 (1996), no. 1, 32-39.

[24] Schneider, Rolf; Schuster, Franz E.; *Rotation equivariant Minkowski valuations,* Int. Math. Res. Not. 2006, Art. ID 72894, 20 pp.

[25] Schuster, Franz E.; Wannerer, Thomas; $GL(n)$ *contravariant Minkowski valuations*, Trans. Amer. Math. Soc. 364 (2012), no. 2, 815-826.

[26] Sherman, Jack; Morrison, Winifred J.; *Adjustment of an Inverse Matrix Corresponding to a Change in One Element of a Given Matrix,* Annals of Mathematical Statistics 21 (1), 1950, 124–127.